Oregon Soergel bimodule workshop

August 2014

Exercises 4.2

Fun skills: Singular Soergel bimodules and the Hecke algebroid (Q1, Q2), Diagrammatics for SSB (Q3, Q4, Q5), perverse sheaves (Q6).

1. Let W be a Weyl group of type A_2, with $S = \{s, t\}$.

a) The Hecke algebroid of W is generated by 8 elements: $\underline{H}_s$ viewed as a map $\emptyset \to \{s\}$, $\underline{H}_s$ viewed as a map $\{s\} \to \emptyset$, $\underline{H}_{sts}$ viewed as a map $\{s\} \to \{s,t\}$, $\underline{H}_{sts}$ viewed as a map $\{s,t\} \to \{s\}$, and the corresponding maps for t instead of s. Write down a minimal set of relations between these 8 maps. (Hint: the quadratic and braid relations in the Hecke algebra become more "local" in the Hecke algebroid.)

b) Identify all the indecomposable singular Soergel bimodules.

2. In a previous exercise, you proved that when $xs > x$, B_{xs} is induced from R^s, i.e. there is some (R, R^s)-bimodule B such that $B_{xs} = B \otimes_{R^s} R$ up to grading shift. Prove this using Williamson's analog to Soergel's Categorification Theorem. Similarly, suppose that w is of maximal length in a coset of W_I, for $I \subset S$ finitary. What similar conclusion can be drawn?

3. Look up the relations for a square of Frobenius extensions in arXiv:1308.5994. They are numbered (1.1) through (1.12). Check that these relations hold in the 2-category of bimodules.

4. Draw the Jones-Wenzl relation in singular Soergel bimodules in type B_2.

5. Consider singular Soergel bimodules in type $\tilde{A}_2$, with $S = \{s, t, u\}$. The objects in this 2-category correspond to proper subsets of S. Let X denote the 1-morphism corresponding to the path $\{s,t\} \to \{s\} \to \{s,u\} \to \{u\} \to \{t,u\}$, and let Y denote the 1-morphism corresponding to the path $\{s,t\} \to \{t\} \to \{t,u\}$.

a) Draw the identity 2-morphisms of X and Y.

b) Construct a morphism ϕ of degree 0 from X to Y. Are there any others?

c) Let ϕ^* denote the vertical flip of ϕ, a map $Y \to X$. Evaluate $\phi \colon \phi^*$.

6. Let Fl_n denote the complex flag variety G/B in type A_{n-1}. In other words, $\mathrm{Fl}_n = \{V^\bullet = (0 \subset V^1 \subset \ldots \subset V^n = \mathbb{C}_n) \mid \dim V^i = i\}$ is the set of flags in a fixed vector space $\mathbb{C}^n$ with basis $\{e_i\}_{1 \le i \le n}$. There is an action of GL_n on Fl_n, and thus an action of the subgroup $S_n \subset GL_n$. The *standard flag* V_{std} is given by $V^k_{\text{std}} = \mathbb{C} \cdot \langle e_i \rangle_{1 \le i \le k}$; its stabilizer is a Borel subgroup B. For any $w \in S_n$, the dimension of $V^k_{\text{std}} \cap w(V_{\text{std}})^l$ is equal to the size of the intersection $\{1, 2, \ldots, k\} \cap \{w(1), w(2), \ldots, w(l)\}$. For any two flags $V^\bullet$ and $W^\bullet$, we say that they are in *relative position* w if $\dim(V^k \cap W^l) = \dim(V^k_{\text{std}} \cap w(V_{\text{std}})^l)$.

a) Show that Fl_n splits into B orbits based on the relative position of a flag with the standard flag, and that this agrees with the usual Bruhat decomposition of G/B. Show that $\mathrm{Fl}_n \times \mathrm{Fl}_n$ splits into G orbits based on the relative position of the two flags. Show that the orbit closure relation agrees with the Bruhat order, in either setting.

Clearly $V^\bullet$ and $W^\bullet$ are in relative position $s_i \in S \subset S_n$ if and only if $V^i \neq W^i$ and $V^k = W^k$ for all $k \neq i$. We say that $V^\bullet$ and $W^\bullet$ are in *relative position* $\overline{s_i}$ if $V^k = W^k$ for all $k \neq i$ (with no condition on V^i and W^i). Let $\underline{w} = s_{i_1} s_{i_2} \ldots s_{i_d}$ be a sequence of simple reflections. The

Bott-Samelson resolution $BS(\underline{w})$ is the space consisting of sequences of flags, ending in the standard flag, and successively in relative position determined by $\underline{w}$:

$$\{(V_i^\bullet)_{i=0}^d \mid V_d^\bullet = V_{\text{std}}^\bullet, \text{ and the pair } (V_{k-1}^\bullet, V_k^\bullet) \text{ is in relative position } \overline{s_{i_k}} \text{ for each } 1 \leq k \leq d\}.$$

It is equipped with a map $\mu\colon BS(\underline{w}) \to \mathrm{Fl}_n$, $\mu((V_i^\bullet)) = V_0^\bullet$.

b) Show that this description of the Bott-Samelson resolution agrees with the one given in lecture.

Set $n = 4$, and let s, t, u denote the simple reflections in S_4 with $su = us$. For an arbitrary flag $W^\bullet$ in each orbit, calculate the fiber $\mu^{-1}(W^\bullet)$ when:

c) $\underline{w} = tt$.

d) $\underline{w} = sts$.

e) $\underline{w} = tsut$.

f) $\underline{w} = sutsu$.

Now, for each of the above cases, construct the table for $\mu_*(\mathbb{C}[\ell(\underline{w})])$. Use these tables (and possibly other calculations) to decompose this pushforward into $\mathcal{I}$ sheaves.

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Supplementary Exercises 4.2

Perverse sheaves and categorical actions:

7. This is a family of possible exercises. We recommend starting with the case $n = 4$ and $\lambda = (2, 2)$, portions of which were discussed in class.

Consider a partition λ of n, and let P_λ be the corresponding parabolic subgroup of GL_n. We will consider P_λ acting on all Grassmannians $\mathbb{G}(k, n)$ for $0 \leq k \leq n$.

a) Classify the P_λ orbits on $\mathbb{G}(k, n)$, and prove that your classification is correct. Compute their dimensions.

b) Find a resolution of singularities of each orbit closure. Compute the fibers over each orbit in this resolution.

c) Construct a table for the pushforward of the IC sheaf (i.e. constant sheaf with shift) on each resolution of singularities. Use this to compute the IC sheaf of the orbit.

Now consider the partial flag variety $\mathbb{F}(k, k + 1, n)$, with its forgetful maps $p\colon \mathbb{F}(k, k + 1, n) \to \mathbb{G}(k, n)$ and $q\colon \mathbb{F}(k, k+1, n) \to \mathbb{G}(k+1, n)$. Let d_k be the dimension of $\mathbb{G}(k, n)$. Define E to be the functor $q_* p^*(\cdot)[d_{k+1} - d_k]$ from perverse sheaves with shifts on $\mathbb{G}(k, n)$ to perverse sheaves with shifts on $\mathbb{G}(k + 1, n)$. Let F be the functor $p_* q^*(\cdot)[d_k - d_{k+1}]$ in the other direction.

d) Compute the table of E and F applied to each IC sheaf. Compute the decomposition into perverse sheaves.

e) Verify that, on the Grothendieck group, $[E]$ and $[F]$ induce an action of $U_v(\mathfrak{sl}_2)$, giving the representation $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_k}$.

f) Find a subcategory of each category of perverse sheaves, preserved under the functors E and F, whose Grothendieck group gives the subrepresentation V_λ. Are there subcategories for other subrepresentations? What about quotient categories, or subquotient categories?

Singular Soergel diagrams:

8. In type A_3, draw the Zamolodchikov relation as an equality of morphisms between singular Soergel bimodules. In fact, both sides of the equation are actually idempotents projecting to B_{w_0}. Using this fact, make an educated guess as to another singular Soergel diagram of degree 0 which is equal to both sides, and which can not be described using ordinary Soergel diagrams. (The proof that these diagrams are all equal is not possible without a special relation between singular diagrams.)