Oregon Soergel bimodule workshop

August 2014

Exercises 4.1

Essential skills: Rouquier complexes and their minimal complexes (Q1, Q2, Q3, Q4), Induced forms (Q5, Q6)

1. Let F_s and F_s^{-1} denote the Rouquier complexes introduced in lectures. Check that $F_sF_s^{-1} \cong R$ in $K^b(R\text{-Bim})$ as sketched in lectures.

2. Compute the terms appearing in the minimal complex of $F_s^{\otimes m}$ for $m \geq 0$. Describe its perverse filtration explicitly.

3. Write down the summands appearing in the minimal complex of $F_sF_uF_tF_sF_u$.

4. Suppose that $m_{st} = 2$. Find explicitly a chain map from F_sF_t to F_tF_s and back. Renormalize your maps such that the composition is the identity chain map.

5. Fix a Soergel bimodule B and consider the two maps $\alpha, \beta : B \to BB_s = B \otimes_R B_s$ given by

$$\alpha(b) := bc_{\mathrm{id}} \quad \text{and} \quad \beta(b) := bc_s.$$

Together, $\alpha(B)$ and $\beta(B)$ span BB_s. Show that β is a morphism of bimodules, whilst α is a morphism of left modules. Find a formula for $\alpha(br)$ for $b \in B$ and $r \in R$.

Suppose that B is equipped with an invariant form $\langle -, - \rangle_B$. Prove that there is a unique invariant form $\langle -, - \rangle_{BB_s}$ on BB_s, which we call the *induced form*, satisfying

$$\langle \alpha(b), \alpha(b') \rangle_{BB_s} = \partial_s(\langle b, b' \rangle_B) \tag{1}$$

$$\langle \alpha(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \text{ and } \langle \beta(b), \alpha(b') \rangle_{BB_s} = \langle b, b' \rangle_B \tag{2}$$

$$\langle \beta(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \alpha_s \tag{3}$$

for all $b, b' \in B$. Show that the intersection form on a Bott-Samelson bimodule agrees with the form induced many times from the canonical form on R.

Now consider $\overline{BB_s}$, with its induced form valued in $\mathbb{R}$. Calculate a matrix for this form in some basis. Prove that the induced form is non-degenerate whenever the original form on $\overline{B}$ is non-degenerate.

6. In this exercise we prove an "easy" case of hard Lefschetz. Assume that B_x is a Soergel bimodule such that hard Lefschetz holds on $\overline{B_x}$. Consider the operator

$$L_\zeta := (\rho \cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_x}(\zeta\rho \cdot -)$$

on B_xB_s. It induces a Lefschetz operator L_ζ on $\overline{B_xB_s}$. (You can equip B_x with an invariant form if you wish, but it won't be important for this exercise.)

a) Let $s \in S$ be such that $xs < s$. Show that $B_xB_s = B_x(1) \oplus B_x(-1)$. (You should be able to give an abstract argument, but in part b) the following fact is useful (see "Singular Soergel bimodules"): there exists an (R, R^s)-bimodule $B_{\overline{x}}$ such that $B_{\overline{x}} \otimes_{R^s} R \cong B_x$.)

b) Rewrite the Lefschetz operator L_z on B_xB_s using a fixed choice of isomorphism $B_xB_s = B_x(1) \oplus B_x(-1)$. Conclude that in the right quotient $\overline{B_xB_s}$, L_ζ has the form

$$\begin{pmatrix} \rho \cdot - & 0 \\ \zeta\gamma & \rho \cdot - \end{pmatrix}.$$

for some non-zero scalar γ. (As above, $\rho \cdot -$ denotes the degree two endomorphism of left multiplication by ρ.) (Hint: Look at the previous exercise.)

c) Conclude that L_ζ satisfies hard Lefschetz on $\overline{B_xB_s}$ if and only if $\zeta \neq 0$.

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Supplementary Exercises 4.1

Rouquier complexes:

7. This exercise is very very computational! (Hint: If you're stuck, look at a paper by Elias-Krasner.) Suppose that $m_{st} = 3$. Find the most general chain map (of degree 0) from $F_s F_t F_s$ to $F_t F_s F_t$ and vice versa (i.e. you should get families of maps given by certain parameters). Compute their composition, an endomorphism of $F_s F_t F_s$. For certain parameters, find a homotopy map between this composition and the identity chain map.

8. This exercise is also very computational.

a) Compute the minimal complex FT of $F_s F_s$, including the differentials.

b) A *shift* is a complex of the form $R(n)\langle k \rangle$, for some grading shift n and homological shift k. Compute all morphisms of complexes from each shift to FT. There should be only two nonzero maps, up to the action of R on morphisms.

c) One of the two maps, which we denote α, is a quasiisomorphism of complexes of R-bimodules. Show that α becomes an isomorphism (i.e. a homotopy equivalence) after tensoring with B_s on the right. (In this situation, we call α an *eigenmap*, and B_s an *eigencomplex*.)

d) Call the other map β. Let Λ_α and Λ_β denote the cones of α and β, respectively. Prove that the tensor product $\Lambda_\alpha \Lambda_\beta$ is nulhomotopic. (This categorifies the relation $(H_s^2 - v^2)(H_s^2 - v^{-2}) = 0$, and proves that FT is *categorically diagonalizable*.)

The embedding theorem:

9. Fix $x \in W$, and work in the category $\mathcal{D}/\mathcal{D}_{<x}$. The local intersection form is a form on the space $\mathrm{Hom}^0(x, BS(\underline{w}))$, where x represents the object B_x in this quotient category. As we have seen, $\mathrm{Hom}^0(x, BS(\underline{w}))$ has a basis given by (upside-down) light leaves of degree 0.

a) Prove that the map $\iota \colon \mathrm{Hom}^0(x, BS(\underline{w})) \to \overline{BS(\underline{w})}$ which sends $\phi \mapsto \overline{\phi(c_{\mathrm{bot}})}$ is injective.

b) Suppose that $\overline{BS(\underline{w})}$ is equipped with a Lefschetz operator given by left multiplication by some $f \in \mathfrak{h}^* \subset R$. Prove that the image of ι consists of primitives in degree $-\ell(x)$.

Positivity and quantum numbers:

10. In this lecture series, we have been assuming that $a_{s,t} = -2\cos\frac{\pi}{m_{st}}$, or in other words, that $a_{s,t} = -(q + q^{-1})$ where $q = e^{\pm\frac{\pi i}{m}}$ is a primitive $2m$-th root of unity. In previous exercises, we have seen that one can set q to be other primitive $2m$-th roots of unity and still obtain an action of W. What positivity considerations will fail if q is set to one of these other roots of unity? For example, what if $m = 53$ and $q = e^{\frac{3\pi i}{53}}$? (Hint: Consider Exercise 4 from the Wednesday exercises.)