Oregon Soergel bimodule workshop
August 2014

Exercises 3

Essential skills: Local intersection forms (Q1, Q2), Elements of Bott-Samelson bimodules (Q3, Q4, Q5), Global intersection forms (Q6, Q7), Lefschetz linear algebra (Q6, Q7, Q8, Q9)

1. a) In type B_2, compute the local intersection pairing of $BS(stst)$ at st, in all degrees.

b) In degree 0, make observations about the definiteness and signature of this form.

c) In type H_2, compute the local intersection pairings of $BS(ststs)$ at sts and s respectively. Any observations about definiteness and signature in degree 0?

d) In type D_4, compute the local intersection pairings of $BS(tuvstuv)$ at tuv, in all degrees. Definiteness and signature in degree 0?

2. In the Temperley-Lieb category with $q = 1$, compute the local intersection form on $\mathrm{Hom}(3,5)$ and on $\mathrm{Hom}(1,5)$. What are the signatures of these forms?

3. Let $f \in \mathfrak{h}^* \in R$ be a linear polynomial. For a general expression $\underline{w}$, find a formula for $fc_{\underline{\varepsilon}}$ in the 01-sequence basis of $BS(\underline{w})$ as a right R-module.

4. In this exercise we find a recursive formula for

$$N_{\underline{w}}(f) := \langle f^{\ell(\underline{w})} c_{\mathrm{bot}}, c_{\mathrm{bot}} \rangle.$$

for any degree two element $f \in R$, acting by left multiplication on $\overline{BS(\underline{w})}$.

a) Find a formula for $N_{\underline{w}}(f)$ in terms of $N_{\underline{w}'}(f)$, over all subexpressions $\underline{w}'$ obtained by omitting a simple reflection from $\underline{w}$.

b) Show that $N_{\underline{w}}(f) = 0$ unless $\underline{w}$ is reduced. (*Hint:* It might help to use the light leaves description of $BS(\underline{w})$ or the decomposition of $BS(\underline{w})$ into indecomposable Soergel bimodules.) Use this to simplify your formula in part (a).

c) Suppose that $\partial_s(f) > 0$ for all $s \in S$. Show that $N_{\underline{w}}(f) > 0$ for $\underline{w}$ reduced. (First prove that $sw > w$ if and only if $\partial_s(wf) > 0$.)

5. Use the 01-basis of $BS(\underline{w})$ and an upper-triangularity argument to prove that the global intersection form is non-degenerate to degree 0.

6. Consider $\overline{(B_s B_s)}$, with the Lefschetz operator

$$L_{a,b} := (a\rho \cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_s}(b\rho \cdot -)$$

for some $a, b \in \mathbb{R}$. For which a, b does the hard Lefschetz property hold? For which a, b do the Hodge-Riemann bilinear relations hold? For which a, b does (HR) hold with the opposite signatures?

7. Now we work with $\overline{BS(sts)}$ when $m_{st} = 3$. Let $\rho \in \mathfrak{h}^*$ satisfy $\partial_s(\rho) = \partial_t(\rho) = 1$. Let L be the degree 2 endomorphism of $\overline{B_s B_t B_s}$ given by left multiplication by ρ.

What is $L^3(c_{\mathrm{bot}})$? What is $\langle c_{\mathrm{bot}}, L^3(c_{\mathrm{bot}}) \rangle$? Find a basis for $\overline{B_s B_t B_s}^{-1}$ (i.e. the elements in degree -1) in the kernel of L^2. Are they orthogonal to $L^2(c_{\mathrm{bot}})$ under the intersection form? Show that the form $(v, w) = \langle v, Lw \rangle$ on this orthogonal subspace of $\overline{B_s B_t B_s}^{-1}$ is negative definite.

Bonus problem: what does the picture look like when restricted to the summand $B_s \overset{\oplus}{\subset} B_s B_t B_s$? What does it look like when restricted to the summand $B_{sts} \overset{\oplus}{\subset} B_s B_t B_s$?

8. Let (V, L_V) and (W, L_W) be *Lefschetz spaces*, i.e. graded vector spaces equipped with a nondegenerate graded bilinear form and a Lefschetz operator. Suppose that $\sigma\colon V \to W(1)$ is a vector space map of degree $+1$, satisfying

- $\sigma L_V = L_W \sigma$,
- $\langle v, L_V v' \rangle_V = \langle \sigma v, \sigma v' \rangle_W$,
- σ is injective from negative degrees.

Suppose that (W, L_W) has (HR). Prove that (V, L_V) has (hL). (Hint: There are two cases, for $v \in V$ of negative degree. Either σv is primitive, or σv is not primitive.) What extra conditions would guarantee that (V, L_V) has (HR), except in degree 0?

9. a) Let (V, L_V) and (W, L_W) be Lefschetz spaces, and suppose that $\sigma\colon V \to W(-d)$ is a vector space map of degree $-d$, satisfying $\sigma L_V = L_W \sigma$. When $d > 0$, prove that the Lefschetz form on W, restricted to the image of σ, is zero.

b) Deduce that the global intersection form on $\overline{B_s B_s}$, restricted to the (canonical) summand $\overline{B_s}(1)$, is zero.

c) By contrast, show that the global intersection form need not restrict to zero on a summand of the form $\overline{B_s}(-1)$. (This summand is non-canonical, so there are multiple choices of inclusion map.)

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Supplementary Exercises 3

Krull-Schmidt categories:

Recall that a *Krull-Schmidt category* is an additive category in which every object is isomorphic to a finite direct sum of indecomposable objects, and an object is indecomposable if and only if its endomorphism ring is local.

10. Some exercises to get used to Krull-Schmidt categories:

a) Show that the Krull-Schmidt theorem holds in Krull-Schmidt categories: any object can be written as a direct sum of indecomposable objects, and this decomposition is unique up to permutation of the factors.

b) *(Idempotent lifting)* Let A be an algebra and $\mathfrak{m} \subset A$ an ideal such that $\mathfrak{m}^2 = 0$. Show that given an idempotent $e \in A/\mathfrak{m}$ there exists an idempotent $\widetilde{e} \in A$ such that $e = \widetilde{e}$ in $A/\mathfrak{m}$. Now prove the same statement assuming only that A is complete with respect to the topology defined by $\mathfrak{m}$.

c) Let $(\mathbb{O}, \mathfrak{m})$ be a complete local ring. Let $\mathcal{C}$ be a Karoubian $\mathbb{O}$-linear additive category such that all hom spaces are finitely generated. Show that $\mathcal{C}$ is Krull-Schmidt. (*Hint:* It might help to first consider the case when $\mathbb{O}$ is a field.)

d) Show that the category of graded modules over a polynomial ring is a Krull-Schmidt category. Conclude that the category of Soergel bimodules is Krull-Schmidt.

e) (*) Let X be an affine variety. When does the Krull-Schmidt theorem hold for vector bundles on X? (Answer: almost never.) Conclude that the Krull-Schmidt theorem fails for ungraded modules over a polynomial ring. (Optional: show that the Krull-Schmidt theorem holds for vector bundles on a projective algebraic variety.)

11. Let $\mathcal{C}$ be a Krull-Schmidt category over an algebraically closed field $\Bbbk$. Show that the multiplicity of B as summand of X is given by the rank of the form

$$\mathrm{Hom}(B, X) \times \mathrm{Hom}(X, B) \to \mathrm{End}(B)/\mathfrak{m}_B.$$

where $\mathfrak{m}_B$ denotes the maximal ideal of $\mathrm{End}(B)$. What is the correct statement for general fields or local rings $\Bbbk$?

Lefschetz linear algebra:

12. Let $H = \bigoplus H^i$ be a finite dimensional graded $\mathbb{R}$-vector space and $L : H^\bullet \to H^{\bullet+2}$ an operator of degree 2. Show that H admits a representation of $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ with $e = L$ and $hx = mx$ for all $x \in H^m$ if and only if L satisfies the hard Lefschetz theorem (i.e. $L^m : H^{-m} \to H^m$ is an isomorphism for all $m \geq 0$).

13. Prove that the Lefschetz decomposition is orthogonal for the Lefschetz form.

14. Suppose that $H = \oplus H^i$ and $W = \oplus W^j$ are finite dimensional graded real vector spaces with forms $\langle -, - \rangle$ and Lefschetz operators L_H and L_W. Suppose that $H^{\mathrm{odd}} = 0$ or $H^{\mathrm{even}} = 0$, that L satisfies the hard Lefschetz theorem on H and that

$$\underline{\dim} W := \sum \dim W^i v^i = (v + v^{-1}) \underline{\dim} H.$$

Show that W satisfies (HR) if and only if the signature of the Lefschetz form $(-,-)^{-i}_{L_W}$ on W^{-i} is equal to the dimension of the primitive subspace $P^{-i+1}_{L_H} \subset H^{-i+1}$ (by convention $P^1_{L_H} = 0$).

15. This question explores Hodge theory for the Grassmannian $H^*(Gr(3,6))$, using a combinatorial model. Let $P(3,6)$ denote the set of partitions which fit inside a 3×3 rectangle (I will describe elements of $P(3,6)$ using Young tableaux). The *degree* of a partition will be -9 plus twice the number of boxes; for example, the partition $(3,1,1)$ has degree $+1$. We say that two partitions are *complimentary* if one can be glued to the 180 degree rotation of the other to obtain the full 3×3 rectangle; for example, $(3,2,0)$ and $(3,1,0)$ are complimentary.

Let H denote the graded vector space with basis $\{v_\lambda\}_{\lambda \in P(3,6)}$. Place a symmetric bilinear form on H, where $\langle v_\lambda, v_\mu \rangle = 1$ when λ and μ are complimentary, and it equals zero otherwise. Place an operator $L \colon H \to H(2)$ on this space, where $L v_\lambda = \sum_\mu v_\mu$ is the sum over partitions $\mu \in P(3,6)$ obtained from λ by adding a single box.

a) Prove that L is a Lefschetz operator.

b) Prove that L has the hard Lefschetz property. Compute a basis of each primitive subspace.

c) Prove that L has the Hodge-Riemann bilinear relations.

Duality and invariant forms

16. Given an R-bimodule B, its dual $\mathbb{D}B$ is defined to be $\mathrm{Hom}_{(-,R)}(B,R)$, the right R-module maps. Clearly $\mathbb{D}^2 = \mathbb{1}$ on any bimodule which is free as a right R-module.

a) Show that $\mathrm{Hom}^0_{(R,R)}(B, \mathbb{D}B)$ is isomorphic to the space of invariant forms on B. If a map $B \to \mathbb{D}B$ is an isomorphism, what does this say about the corresponding invariant form?

b) What is $\mathbb{D}B_s$? What about $\mathbb{D}BS(\underline{w})$?

c) Show (by definition) that $\mathbb{D}B_w \cong B_w$ for all $w \in W$, and therefore there exists a nondegenerate invariant form on B_w.

d) If the Soergel conjecture holds, show that any non-zero invariant form on B_w is nondegenerate. Show that the global intersection form on $BS(\underline{w})$ for a reduced expression restricts to a nonzero form on B_w.