Oregon Soergel bimodule workshop

August 2014

Exercises 2.2

Essential skills: Constructing light leaves and double leaves (Q1, Q2, Q3), understanding the Zamolodchikov relations (Q4, Q5), the light leaves basis theorem (Q6, Q7). For more practice in 2 colors and category $\mathcal{O}$, see supplementary exercises.

1. Describe all light leaves maps from $ss\ldots s$ (m times).

2. The diagram $ss \to s$ of degree $+1$, which is a horizontal reflection of the light leaf for 01, is not a light leaf. Rewrite this morphism as an R-linear combination of double leaves.

3. Let W be of type A_7, and let $\underline{w}$ be the reduced expression

$$\underline{w} = 1357246352461357.$$

a) Show that

$$\underline{e} = 1111010110100000$$

is the unique subexpression with defect zero and terminus

$$w_I = 13435437.$$

(Note that w_I is the longest element of the parabolic subgroup for $I = \{1, 3, 4, 5, 7\}$.)

b) Draw the corresponding light leaf.

c) Take this light leaf, and precompose it with the upside-down version of itself, to obtain a morphism $\underline{w_I} \to \underline{w} \to \underline{w_I}$. Compute this morphism, modulo terms lower than w_I. (A lengthy calculation, this is a supplemental exercise. The answer should be multiplication by 2!)

4. Let $S = \{s, t, u\}$ be type A_3. Let $\underline{w} = tstuts$ and let $\underline{y} = utstut$ be two expressions for the longest element $w_0 \in W$. There are (essentially) two paths from $\underline{w}$ to $\underline{y}$ in the reduced expression graph of w_0. Find a reasonably quick proof that the two corresponding morphisms of Bott-Samelson bimodules are not equal. (Extra Credit: find the lower terms which express the difference of these two morphisms.)

5. Let $S = \{s, t, u\}$ be type B_3, with $m_{st} = 3$ and $m_{tu} = 4$. The "miraculous" reduced expression is $\underline{w} = stustsutu$. Draw the Zamolodchikov relation.

6. Fix $\underline{w}$ arbitrary, and $\underline{x}$ reduced. Let $E(\underline{w}, x)$ denote the set of light leaves for subexpressions of $\underline{w}$ which terminate in x, living inside $\mathrm{Hom}(\underline{w}, \underline{x})$. Use localization and the Bruhat path dominance order to prove that the images in $\mathbb{BS}\mathrm{Bim}$ of the light leaves maps in $E(\underline{w}, x)$ are all linearly independent.

7. Show that the functor from $\mathcal{D}$ to $\mathbb{BS}\mathrm{Bim}$ is an equivalence of categories, assuming that double leaves form a basis for morphisms in $\mathcal{D}$.

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Supplementary Exercises 2.2

Practice in two colors:

8. Verify that the two-color relations (the "dot" relation and the "associativity" relation) imply the "idempotent decomposition" relation.

9. Let $m_{st} = m < \infty$. For $k > 0$, let $\underline{w} = stst \dots st$ of length $2(m+k)$. What is the dimension of $\mathrm{Hom}(BS(\underline{w}), R)$ in degree $-2k$? Draw a light leaf map in that degree. Now draw several different graphs realizing the same morphism.

Category $\mathcal{O}$:

10. These questions are about category $\mathcal{O}$ for $\mathfrak{sl}_2$.

a) Find a change of basis to check directly that $\Delta(5) \otimes L(1) \cong \Delta(4) \oplus \Delta(6)$.

b) Find projective resolutions of $\Delta(0)$ and $\Delta(-2)$.

c) Find a projective resolution of $L(0)$ and $L(-2)$.

d) Find a projective resolution of $\nabla(0)$ and $\nabla(-2)$.

e) After applying the Soergel functor, these resolutions are sent to complexes of Soergel modules. Write down these complexes. How can you deduce what the differentials are?

11. In this exercise we look at the effect of translation functors on category $\mathcal{O}$, and see that they are easily understood on Verma modules.

i) Let $\lambda \in \mathfrak{h}^*$ be an arbitrary weight, and let V be a finite dimensional representation of $\mathfrak{g}$. Show that $\Delta(\lambda) \otimes V$ has a Verma flag; that is, that there exists a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = \Delta(\lambda) \otimes V$$

such that $F_i/F_{i-1} \cong \Delta(\mu_i)$ for some $\mu_i \in \mathfrak{h}^*$. What can you say about the multiset $\{\mu_i\}$?

ii) Now suppose that $\lambda, \mu \in \mathfrak{h}^*$ are such that $\lambda + \rho, \mu + \rho$ are dominant, and such that $\lambda - \mu \in \mathbb{Z}R$. Show that $T_\lambda^\mu(\Delta(w \cdot \lambda)) \cong \Delta(w \cdot \mu)$. Conclude that T_λ^μ gives an equivalence $\mathcal{O}_\lambda \to \mathcal{O}_\mu$ if $\lambda + \rho$ and $\mu + \rho$ are strictly dominant. Moreover, show that $T_\lambda^\mu \circ T_\nu^\lambda \cong T_\nu^\mu$ whenever $\mu + \rho, \lambda + \rho, \nu + \rho$ are all strictly dominant.

iii) Now suppose that λ is integral and that $\lambda + \rho$ is dominant. Show we have an isomorphism

$$[\mathcal{O}_\lambda] \to \mathbb{Z}We_\lambda : [\Delta(w \cdot \lambda)] \mapsto e_\lambda \cdot w$$

where $e_\lambda = \sum_{x \in \mathrm{Stab}_W(\lambda+\rho)} x$.

iv) Let λ, μ be as above. In addition, assume that λ, μ are integral, that λ is regular (i.e. $\lambda + \rho$ is strictly dominant) and the μ is sub-regular (i.e. $e_\mu = (1+s)$ for some $s \in S$). Show that we have a commutative diagram

$$\begin{tikzcd}
[\mathcal{O}_\lambda] \arrow[r, "T_\lambda^\mu"] \arrow[d, "\sim"] & [\mathcal{O}_\mu] \arrow[r, "T_\mu^\lambda"] \arrow[d, "\sim"] & [\mathcal{O}_\lambda] \arrow[d, "\sim"] \\
\mathbb{Z}W \arrow[r, "\cdot(1+s)"] & \mathbb{Z}W(1+s) \arrow[r, "\text{inclusion}"] & \mathbb{Z}W
\end{tikzcd}$$

(the vertical isomorphisms are those of the previous exercise).

v) (Optional) Can you give similar descriptions for more general weights? (I.e. non integral, or with e_λ more complicated?)

12. a) Let C be a finite dimensional graded algebra, and P a (non-graded) projective (resp. simple) module. Show that P admits a graded lift.

b) Show that $\overline{B_x}$ is indecomposable as a graded R-module if and only if it is indecomposable as an ungraded R-module.

Light leaves and indecomposables:

These exercises were written hastily, perform at your own risk.

13. Let $\underline{w}$ and $\underline{x}$ be rexes. We work modulo terms lower than x. We have seen that the coefficient of H_x inside $\underline{H}(\underline{w})$ describes the graded rank of $\mathrm{Hom}(BS(\underline{w}), B_x)$ modulo lower terms, and that light leaves for $\underline{w}$ with terminus x give a basis for this space (as a right R-module). Let e_w denote the idempotent in $\mathrm{End}(BS(\underline{w}))$ which picks out the indecomposable B_w. Let the *x-kernel of $\underline{w}$* be those linear combinations of light leaves with terminus x which vanish after precomposition with e_w. Then $\mathrm{Hom}(B_w, B_x)$ is precisely $\mathrm{Hom}(BS(\underline{w}), B_x)$ modulo the x-kernel, modulo lower terms.

a) Justify that the graded rank of $\mathrm{Hom}(BS(\underline{w}), B_x)$ modulo lower terms should agree with the coefficient of H_x in the character of $BS(\underline{w})$.

b) Assuming that $[B_w] = \underline{H}_w$, justify that the graded rank of $\mathrm{Hom}(B_w, B_x)$ modulo lower terms should agree with the coefficient $h_{x,w}$ of H_x in $\underline{H}_w$.

c) Let $m_{st} = 3$. Compute the x-kernel of the rex $\underline{sts}$, for each $x \leq sts$. Do the graded ranks agree with your expectations?

14. Recall that $\underline{H}_w \underline{H}_s = \underline{H}_{ws} \sum_y \mu(y, w, s) \underline{H}_y$ for various integers $\mu(y, w, s)$.

a) Using the inductive algorithm, prove that $\mu(y, w, s)$ is zero unless $ys < y$. When $ys < y$, prove that $\mu(y, w, s)$ is equal to the coefficient of v^1 in $h_{y,w}$.

b) Assuming that one knows the y-kernel of $\underline{w}$, construct a diagrammatic basis of $\mathrm{Hom}^0(B_w B_s, B_y)$. Use symbols to denote e_w and e_y. (Hint: How does the light leaf construction connect degree $+1$ maps from $\underline{w}$ and degree $+0$ maps from $\underline{ws}$?)