Oregon Soergel bimodule workshop
August 2014

Exercises 2.1

Essential skills: Computing with diagrams (Q1, Q2), diagrammatic arguments (Q3), idempotent decompositions (Q2, Q5), Temperley-Lieb algebra and Jones-Wenzl projectors (Q6), localization (Q7, Q8)

1. Check that the one color relations hold in Soergel bimodules.

Remark. In an additive category, in order to demonstrate morphism-theoretically that $X \cong M \oplus N$, one must provide morphisms

$$p_M \colon X \to M, \qquad i_M \colon M \to X, \qquad p_N \colon X \to N, \qquad i_N \colon N \to X,$$

which satisfy the following relations:

$$\begin{aligned} p_M i_M &= \mathbb{1}_M, \\ p_N i_N &= \mathbb{1}_N, \\ p_M i_N &= 0, \\ p_N i_M &= 0, \\ \mathbb{1}_X &= i_M p_M + i_N p_N. \end{aligned}$$

Comprehend this fact. The final equation decomposes the identity of X into orthogonal idempotents.

2. Show that $B_s B_s \cong B_s(1) \oplus B_s(-1)$ by following the rubric of the remark above.

3. a) Consider a one-color Soergel diagram without polynomials, viewed as a graph (with boundary) having only trivalent and univalent vertices. Prove that any two trees with the same boundary are equal. Prove that any connected graph which is not a tree evaluates to zero.

b) Prove that any universal morphism (in many colors) with empty boundary is equal to a polynomial. (Hint: use induction on the number of connected components.)

4. The graded rank (as a free right R-module) of the following Hom spaces was computed in the previous supplemental exercises to be

a) v^2 for $\mathrm{Hom}(B_s, B_t)$,

b) $v^{-1} + 2v + v^3$ for $\mathrm{Hom}(B_s B_s, B_s)$,

c) $1 + 2v^2 + v^4$ for $\mathrm{Hom}(B_s, B_s B_t B_s)$.

Now construct diagrammatic bases for these spaces. (Hint: They only use universal diagrams.)

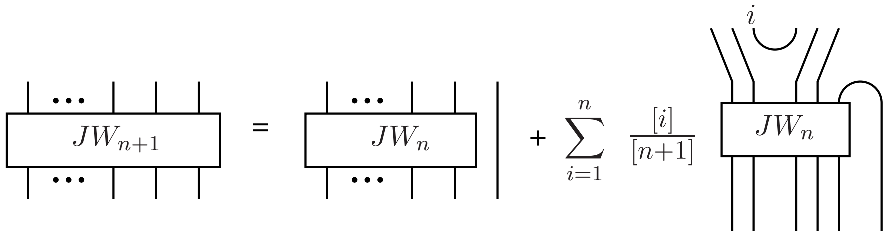
5. a) Write down the two-color relations when $m_{st} = 2$. Prove that $B_s B_t \cong B_t B_s$ by constructing inverse isomorphisms.

b) Write down the two-color relations when $m_{st} = 3$. Prove that $B_s B_t B_s \cong X \oplus B_s$, where X is the image of an idempotent constructed using two 6-valent vertices, by following the rubric of the remark above.

c) (Still with $m_{st} = 3$.) One similarly has $B_t B_s B_t \cong Y \oplus B_t$. Prove that X is isomorphic to Y. (Extra credit: Extend the remark above to a rubric which describes when two summands of different objects are isomorphic.)

6. Let TL_n be the Temperley-Lieb algebra with n strands, where a circle evaluates to $-[2] = -(q+q^{-1}) \in \mathbb{Z}[q,q^{-1}]$. Show that the space of all elements killed by caps above (resp. cups below) is at most one-dimensional, and show that these spaces agree. (By constructing the Jones-Wenzl projector, one proves that this space is precisely one-dimensional.)

The Jones-Wenzl projector $JW_n \in TL_n$ is uniquely specified in this one-dimensional kernel by the fact that the coefficient of the identity is 1. Verify the following recursive formula.



7. Let $2c_s = \alpha_s \otimes 1 + 1 \otimes \alpha_s$ and $2d_s = \alpha_s \otimes 1 - 1 \otimes \alpha_s$ inside B_s. When s is understood, write $q_0 = c_s$ and $q_1 = d_s$. Let $\underline{w}$ have length d. For a subsequence $\underline{e} \subset \underline{w}$, write $q_{\underline{e}}$ for the tensor product $q_{e_1} q_{e_2} \cdots q_{e_d} \in B_{s_1} B_{s_2} \cdots B_{s_d}$. Prove that $\{q_{\underline{e}}\}_{\underline{e} \subset \underline{w}}$ is linearly independent in the degree $+d$ part of $BS(\underline{w})$. Is it a basis?

8. (Assumes knowledge of the support of a coherent sheaf.) For $w \in W$, let $\mathrm{Gr}_w = \{(w(v), v\} \subset \mathfrak{h} \times \mathfrak{h}$. Let $w_1, w_2, \ldots$ be an enumeration of the elements of W, and let B be an R-bimodule. Suppose there exists a filtration $0 \subset B^1 \subset \ldots \subset B^m = B$ such that $B^i / B^{i-1} \cong \oplus R_{w_i}^{\oplus n_i}$. Show that B^i is equal to the submodule of B consisting of sections with support on the subvariety $\cup_{j=1}^{i} \mathrm{Gr}_{w_j}$. Deduce that a standard filtration on a Soergel bimodule is unique and is preserved by all morphisms. (Hint: the support of any nonzero element of R_x is Gr_x.)

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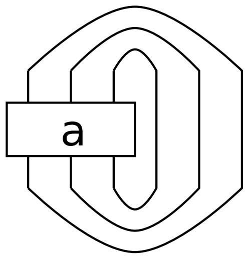
Supplementary Exercises 2.1

9. Why does the general polynomial forcing relation follow from the case of linear polynomials? Give a quick and elegant argument.

10. Consider a universal diagram in two colors, whose boundary alternates $B_sB_tB_sB_t \dots B_sB_t$ when read clockwise. Prove that the minimal degree of any nonzero universal diagram is 2, and that all such universal diagrams are deformation retracts of colored Temperley-Lieb diagrams. (Hint: what characterizes universal diagrams in the image of the functor from Temperley-Lieb? What happens when you try to connect two different connected components of the same color?)

11. The *trace* of an element $a \in TL_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the closed diagram below. Calculate the trace of JW_n (hint: use induction).



Suppose that q is a primitive $2(n+1)$-th root of unity. What is the trace of JW_n? What do you get when you rotate JW_n by one strand?

12. a) Suppose that $\underline{w}$ is a reduced expression. How many copies of Q_w appear in the localization of $BS(\underline{w})$? Does this depend on the reduced expression?

b) Suppose that $\underline{w}$ and $\underline{w'}$ are reduced expressions which differ by a single braid relation. Consider a $2m_{st}$ valent vertex, viewed as a map $BS(\underline{w}) \to BS(\underline{w'})$. Now apply the localization functor to this map. After restriction of this map to $Q_w \to Q_w$, do you get an isomorphism?

c) If you compose two $2m_{st}$-valent vertices $BS(\underline{w}) \to BS(\underline{w'}) \to BS(\underline{w})$, what can you say about the localized restriction to $Q_w \to Q_w$? Is it the identity map?

d) Now begin at $\underline{w}$, and apply an arbitrary chain of braid relations, viewed as $2m_{st}$-valent vertices, to get from $BS(\underline{w})$ to $BS(\underline{w})$. A priori, what can you say about the localized restriction to $Q_w \to Q_w$?

Constructing idempotents

13. In this exercise, we work in type B_2, so that $m_{st} = 4$, and we use a non-symmetric Cartan matrix where $a_{s,t} = -1$ and $a_{t,s} = -2$.

a) In type B_2, write $\underline{H}_s\underline{H}_t\underline{H}_s\underline{H}_t$ as a sum of KL basis elements. How do you expect $B_sB_tB_sB_t$ to decompose?

b) Calculate the graded rank of $\mathrm{Hom}(B_sB_t, B_sB_tB_sB_t)$. Compute a diagrammatic basis of maps in degree 0 (you should have found it to be a 2-dimensional space).

c) Calculate the graded rank of $\mathrm{Hom}(B_sB_tB_sB_t, B_sB_t)$. Compute a diagrammatic basis of maps in degree 0. Why is this really easy, given the last part?

d) Calculate the graded rank of $\mathrm{End}(B_s B_t)$ and deduce that the only degree zero map is the identity.

e) Therefore, one can construct a 2×2 matrix given by composing a map $B_s B_t \to B_s B_t B_s B_t$ of degree 0 with a map $B_s B_t B_s B_t \to B_s B_t$ of degree 0, and computing the coefficient of the identity. This is called a *local intersection form*; one thinks of it as a bilinear form on $\mathrm{Hom}(B_s B_t B_s B_t, B_s B_t)$... How? Compute this matrix.

f) Whenever two maps pair under the local intersection form to the value 1, one can construct an idempotent in $\mathrm{End}(B_s B_t B_s B_t)$ which factors through $B_s B_t$. Whenever one has dual bases under the local intersection form, the corresponding idempotents will be orthogonal. Find dual bases and compute these orthogonal idempotents.

g) You have just proven that $B_s B_t$ occurs as a summand inside $B_s B_t B_s B_t$ precisely 2 times. Can there be any other summands besides B_{stst}? Why or why not?

h) Suppose that we work in characteristic 2. How many times does $B_s B_t$ occur as a summand inside $B_s B_t B_s B_t$?

14. What happens if you repeat Q13 in type H_2? One has $m_{st} = 5$, and $a_{s,t} = a_{t,s} = -\phi$, the (negative) golden ratio.

15. If you want more exercise, repeat Q13 in type H_2, except with the goal of decomposing $B_s B_t B_s B_t B_s$.

16. Let V be the standard representation of $\mathfrak{sl}_2$. Compute the decomposition of $V \otimes V \otimes V$ into direct summands, by constructing an idempotent decomposition of the identity. Does this remind you of any previous exercises? What happens when q is an 8-th root of unity?