Oregon Soergel bimodule workshop

August 2014

Exercises 1.2

Essential skills: Deodhar defect formula (Q1), the pairing on $\mathbf{H}$ (Q2), understanding diagrammatics (Q3, Q4, Q5), playing with elements of Bott-Samelson bimodules (Q6, Q7)

1. a) Use the Deodhar defect formula to compute $\underline{H}_s \underline{H}_s \underline{H}_s$ in the standard basis.

b) Let s, t, u denote three distinct simple reflections. Use the Deodhar defect formula to compute $\underline{H}_s \underline{H}_t \underline{H}_u$. Is this product equal to $\underline{H}_{stu}$?

c) Let s and t be distinct simple reflections. What is $\varepsilon(\underline{H}_s \underline{H}_t \underline{H}_s)$?

d) Let $\{s, t, u, v\}$ be the simple reflections in type D_4, where s, u, v all commute. Compute the product $\underline{H}(\underline{w})$ for the reduced expression $\underline{w} = suvtsuv$.

2. Compute the pairing $(\underline{H}_s \underline{H}_t \underline{H}_s, \underline{H}_s)$ in two different ways.

a) Use biadjunction and the quadratic relation to express this pairing in terms of $\varepsilon(\underline{H}_t \underline{H}_s)$.

b) Use the Deodhar defect formula on both sides, and the "self-dual orthogonality" of the standard basis.

3. Look up the definition of a *Frobenius object* in a monoidal category (on wikipedia).

a) Express this definition diagrammatically.

b) Suppose that $A \subset B$ is a Frobenius extension. Using 1-manifold diagrams for induction and restriction bimodules, show that $B \otimes_A B$ is a Frobenius object in the category of B-bimodules.

4. Let $A = \mathbb{R}[x]/(x^2)$ be an object in the monoidal category of $\mathbb{R}$-vector spaces. Let $\cap \colon A \otimes A \to \mathbb{R}$ denote the map which sends $f \otimes g$ to the coefficient of x in fg. Let $\cup \colon \mathbb{R} \to A \otimes A$ denote the map which sends 1 to $x \otimes 1 + 1 \otimes x$.

a) We wish to encode these maps diagrammatically, drawing $\cap$ as a cap and $\cup$ as a cup. Justify this diagrammatic notation, by checking the biadjointness/isotopy relations.

b) Draw a sequence of nested circles, as in an archery target. Evaluate this morphism.

5. This question deals with the universal presentation of Ω_G for a group G.

a) Draw the generating morphisms corresponding to the pair $g \circ g^{-1}$. Draw the inverse relations.

b) Draw the associator relation for the triple $g \circ g^{-1} \circ g$.

c) Prove that g and g^{-1} are biadjoint.

6. a) Construct a map $B_s B_t \to R \otimes_{R^{s,t}} R(2)$ sending $1 \otimes 1 \otimes 1 \mapsto 1 \otimes 1$, when $m_{s,t} = 2$.

b) Why is there no such map when $m_{s,t} > 2$?

c) Confirm the decomposition $B_s B_t B_s \cong R \otimes_{R^{s,t}} R(3) \oplus B_s$ given in class, when $m_{s,t} = 3$.

d) When $m_{s,t} = 4$, how would you expect $B_s B_t B_s B_t$ to decompose?

7. Let $c_s = \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}$ and $c_1 = 1 \otimes 1$ denote certain elements of B_s. Show that $\{c_s, c_1\}$ form a basis of B_s as a right R-module. For any $f \in R$, find a nice formula for $f c_s$ in terms of this basis.

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Supplementary Exercises 1.2

Hecke algebras:

8. Some more questions from lecture, dealing with the standard trace and standard pairing.

a) Compute $\varepsilon(H_x H_y)$. When is it non-zero?

b) Show that $\varepsilon(ab) = \varepsilon(ba)$.

c) Show that the standard basis is dual to the bar involution of the standard basis for the standard pairing.

d) Show that the KL basis is graded orthonormal for the standard pairing.

e) Show that $\underline{H}_s$ is self-biadjoint.

Deodhar defect formula:

9. Prove the Deodhar defect formula.

10. Let $\underline{w} = s_1 \dots s_m$ denote an expression. We write $x \leq \underline{w}$ if there exists a subexpression $\mathbf{e}$ of $\underline{w}$ with $x = \underline{w}^{\mathbf{e}}$. Given two subexpressions $\mathbf{e}, \mathbf{e}'$ of $\underline{w}$ let $x_0, x_1, \dots$ and $x'_0, x'_1, \dots$ be their Bruhat strolls (e.g. $x_i := s_1^{e_1} \dots s_i^{e_i}$). We define the *path dominance order* on subexpressions by saying that $\mathbf{e} \leq \mathbf{e}'$ if $x_i \leq x'_i$ for $1 \leq i \leq \ell(\underline{w})$. Show that for any $x \leq \underline{w}$ there is a unique subexpression $\mathbf{e}$ of $\underline{w}$, the *canonical subexpression*, which is characterized by the following equivalent conditions:

a) $\mathbf{e} \leq \mathbf{e}'$ for any subexpression $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$ (i.e. $\mathbf{e}$ is the unique minimal element in the path dominance order).

b) $\mathbf{e}$ has no D's in its UD labelling.

c) $\mathbf{e}$ is of maximal defect amongst all subexpressions $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$.

(If you know about Bott-Samelson resolutions: What geometric fact does the existence of $\mathbf{e}$ correspond to?) (Do you think there is a unique maximal element in the path dominance order?)

Diagrammatics:

11. In class, I fixed a biadjunction between E and $E^\vee$, and a biadjunction between F and $F^\vee$. I demonstrated two ways to take a 2-morphism $\beta \colon E \to F$ and return a 2-morphism $F^\vee \to E^\vee$, known as the right mate $\beta^\vee$ and the left mate ${}^\vee\beta$. One can think about these as "twisting" or "rotating" β by 180 degrees to the right or to the left. Visualize what it would mean to twist β by 360 degrees to the right, yielding another 2-morphism $\beta^{\vee\vee} \colon E \to F$. Verify that β is cyclic, i.e. $\beta^\vee = {}^\vee\beta$, if and only if $\beta = \beta^{\vee\vee}$.

12. Suppose that B is an object in a monoidal category with biadjoints, and $\phi \colon B \otimes B \otimes B \to \mathbb{1}$ is a cyclic morphism. What should it mean to "rotate" ϕ by 120 degrees? Suppose that $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{C}$. What can you say about the 120 degree rotation of ϕ, vis a vis ϕ?

Soergel Hom Formula:

13. You already have enough “building block morphisms” to solve these problems.

a) Compute the size (i.e. graded rank) of the Hom space $\mathrm{Hom}(B_s, B_t)$. Find a generating set of morphisms, and indicate how these morphisms factor.

b) Compute the size of the Hom space $\mathrm{Hom}(B_s, B_s B_t B_s)$ in degree zero. Find a generating set of morphisms, and indicate how these morphisms factor.

c) Compose a morphism of minimal degree $B_s \to B_s B_s$ with one of minimal degree $B_s B_s \to B_s$. What is the resulting map? Show this by a general argument, and then by direct computation.