POSITIVITY IN COMBINATORIAL ALGEBRAIC GEOMETRY

Exercises on Matroids and Tropical Linear Spaces

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(1) Linear matroids as matroids of subspaces. Let A and B be two configurations of n vectors in $\mathbb{R}^r$ each. Having chosen a coordinate system, we may regard them as the columns of two $r \times n$ matrices, which we also call A and B. If $\operatorname{rowspace}(A) = \operatorname{rowspace}(B)$, prove that A and B have the same matroid.

(2) An example of a matroid and a tropical linear space.
Let M be the matroid of the root system $A_3 = \{e_i - e_j \mid 1 \leq i \neq j \leq 4\}$. Write down:
- the circuits of M,
- the flats of M,
- the lattice of flats of M, and
- the order complex of the reduced lattice of flats of M.

Draw the Bergman fan $\operatorname{Trop}'(M)$, and describe its homotopy type.

Compute:
- the characteristic polynomial of A_3,
- the Tutte polynomial of A_3,
- the h-polynomial of A_3, and
- the number of regions of the braid arrangement $x_i = x_j$ (for $1 \leq i < j \leq 4$) in $\mathbb{R}^4$.

(3) Recovering a matroid from its Bergman fan
How does one recover the matroid M on E from the Bergman fan $\operatorname{Trop}(M) \in \overline{\mathbb{R}}^E$?

(4) Graphical matroids are linear.
Let $G = (V, E)$ be an undirected graph and $M(G)$ be its matroid. Consider the vector space $\mathbb{R}^V$, and let x_v be the v-th unit vector for each vertex $v \in V$. To each edge $e = \{a, b\}$ of the graph, assign the vector $v_e = x_a - x_b$. Prove that the matroid of the vector configuration $\{v_e : e \in E\}$ is isomorphic to $M(G)$.

(5) The root system A_{n-1} and the space of phylogenetic trees.
Consider the matroid of the root system $A_{n-1} = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$.
- Compute the Tutte polynomial of A_{n-1}.
(This is best expressed in terms of the generating function $\sum_{n\geq 1} T_{A_{n-1}}(x,y) z^n/n!$.)
- Prove that $\operatorname{Trop}(A_{n-1})$ can be regarded as the space of phylogenetic n-trees, as follows.
An *equidistant n-tree* T is a rooted tree with n leaves labelled $1, \ldots, n$, and lengths (real numbers) assigned to each edge so that the path from the root to any leaf has the same length. The internal edges are forced to have positive lengths, while the edges incident to a leaf are allowed to have negative lengths. To each equidistant n-tree T we assign a distance vector $d_T \in \mathbb{R}^{\binom{[n]}{2}}$: the distance $(d_T)_{ij}$ is equal to the length of the path joining leaves i and j in T.
For a point $w \in \mathbb{R}^{\binom{n}{2}}$, prove that the following are equivalent:
  (a) $-w$ is in $\operatorname{Trop}(A_{n-1})$.
  (b) For all distinct $i, j, k \in [n]$, $\max\{w_{ij}, w_{jk}, w_{ik}\}$ is achieved twice.
  (c) w is the distance function of an equidistant n-tree.