Rank 3 Characterizations of Classical Geometries*

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Ever since Higman's ground-breaking work [2], there have been many rank 3 characterizations of the classical symplectic, unitary and orthogonal groups and geometries. The purpose of this paper is to add another such characterization to that pile. Take a finite symplectic, unitary or orthogonal geometry having a totally singular (projective) plane. Let k be the number of singular points $\neq x$ orthogonal to a given singular point x; let l be the number of singular points not orthogonal to x. Finally, let G be a rank 3 permutation group such that the stabilizer of a point has orbits of lengths 1, k, and l. We will show that G can be regarded as an automorphism group of the given geometry acting on the singular points.

This result will be deduced from general theorems which even give some information when singular lines but no singular planes exist. The main idea of the proof is very elementary, essentially the same as that of Kantor [6]. The actual identification of the geometries is made using the theorem of Buekenhout–Shult [1].

We refer to Higman [2] for the relevant background concerning rank 3 groups, and to Higman-McLaughlin [5], Perin [7], and Stark [8] for what is known about rank 3 subgroups of symplectic, unitary, and orthogonal groups.

The precise statements are as follows.

THEOREM 1. *Let G be a primitive rank 3 permutation group on a finite set S. For $x \in S$, let G_x have three orbits of lengths 1, $q\gamma$, and $q^2r\delta$, where q, γ, r, and δ are positive integers satisfying: q and r are powers of a prime $p \nmid \gamma\delta$, $q > 1$, and either*

(1) $(p + q\delta)r \geqslant q\gamma$ *and* $(\gamma, \delta) = 1$

or

(2) $q = p = 2, \delta = 1$.

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Then S can be identified as one of the following geometric objects, in such a way that G acts as an automorphism group of the geometry: the set of singular points of a symplectic, unitary or orthogonal geometry, or the set of points of a generalized quadrangle (with parameters q, $\gamma - 1$).

See Buekenhout–Shult [1] or Higman [4] for brief discussions of generalized quadrangles. Recall that the parameters of a generalized quadrangle are s, t if there are $s + 1$ points per line and $t + 1$ lines per point.

We remark that the proof in case (1) is easier than in case (2). "Most" symplectic, unitary, and orthogonal groups are characterized by Theorem 1 (for the case of $PGO^+(6, q)$, see Corollary 2):

COROLLARY 1. *Let $q > 1$ be a prime power. Let G be a rank 3 group with subdegrees*

(i) $1, q(q^{n-2} - 1)/(q - 1), q^{n-1}$;

(ii) $1, q^2(q^{n-1} - (-1)^{n-1})(q^{n-2} - (-1)^{n-2})/(q^2 - 1), q^{2n-1}$; *or*

(iii) $1, q(q^{m-2} - \epsilon)(q^{m-1} + \epsilon)/(q - 1), q^{2m-2}$,

where $n \geqslant 6$, $m \geqslant 4$, and $\epsilon = \pm 1$ in (iii). Then G can be regarded as an automorphism group of a symplectic, unitary, or orthogonal geometry, acting on the set of singular points.

Case (i) of Corollary 1 with $n = 4$ and arbitrary q, or $n \geqslant 4$ and prime-power q, has been considered by Higman [2] and Tsuzuku [9]. They obtained a similar result by assuming the existence of at least q elations (alias transvections) with a given center.

COROLLARY 2. *A rank 3 group G with subdegrees $1, q(q + 1)^2, q^4$, for a prime power $q > 1$, is a subgroup of* Aut $PSL(4, q)$, *and can be regarded as acting on the set of lines of $PG(3, q)$. Moreover, if $q > 2$ then G contains $PSL(4, q)$; if $q = 2$ then G can also be A_7 or S_7.*

Recall that Aut $PSL(4, q) \cong$ Aut $PGO^+(6, q)$. Higher dimensional analogues of Corollary 2 have been obtained by Higman [3] using entirely different methods.

We will use the following notation. G is a rank 3 permutation group on S. For $x \in S$, the orbits of G_x are $\{x\}$, $\Gamma(x)$, $\Delta(x)$, with $|\Gamma(x)| = q\gamma$ and $|\Delta(x)| = q^2 r\delta$, where $q > 1$ and r are powers of a prime $p \nmid \gamma\delta$. Let $x^\perp = \{x\} \cup \Gamma(x)$. If $y \in \Gamma(x)$, the *singular line* xy is defined by

$$xy = \bigcap \{w^\perp \mid x, y \in w^\perp\} = \bigcap \{w^\perp \mid w \in x^\perp \cap y^\perp\}. \tag{3}$$

A singular line is uniquely determined by any two of its points; all singular lines have the same size $h + 1$ (Higman [2]). Since x is in $q\gamma/h$ lines, of which $\lambda + 1 = |\Gamma(x) \cap y^\perp|$ are in $y^\perp$ (where $\lambda = |\Gamma(x) \cap \Gamma(y)|$ and $y \in \Gamma(x)$), it follows that

$$h \mid (q\gamma, \lambda + 1). \tag{4}$$

Clearly $|\Delta(x) \cap \Gamma(y)| = q\gamma - \lambda - 1$ if $y \in \Gamma(x)$. Write $\mu = |\Gamma(x) \cap \Gamma(w)|$ if $w \in \Delta(x)$. Then $q\gamma(q\gamma - \lambda - 1) = (q^2 r\delta)\mu$. Since G is primitive, $\mu \neq 0$ (Higman [2]).

Now assume (1) or (2). Then $(\gamma, qr\delta) = 1$, so $q\gamma - \lambda - 1 = qr\delta\tau$ and $\mu = \gamma\tau$ for some integer τ. If (1) holds, then $qr\delta\tau < q\gamma \leqslant 2qr\delta$, so $\tau = 1$ and

$$q\gamma - \lambda - 1 = qr\delta \quad \text{and} \quad \mu = \gamma. \tag{5}$$

If (2) holds then $\gamma\tau = \mu \leqslant q\gamma = 2\gamma$, so (5) again is satisfied unless $\mu = q\gamma$. But $\mu = q\gamma$ implies that G is imprimitive (Higman [2]). This proves (5). We can now substitute (5) for parts of (1) and (2):

THEOREM 2. *The conclusions of Theorem* 1 *hold if* (1) *and* (2) *are replaced by*

$$(p + q\delta)r \geqslant q\gamma \quad \textit{and} \quad (5)\ \textit{holds}, \tag{1$'$}$$

and

$$q = p,\ \delta = 1 \quad \textit{and} \quad (5)\ \textit{holds}. \tag{2$'$}$$

In the remainder of the proof, we will assume (1′) or (2′).

Each orbit on $\Delta(x)$ of a Sylow p-subgroup Q of G_x has length $\geqslant q^2 r$. Since $p \nmid \gamma$, we may assume that $|Q : Q_y| = q$ and $P = Q_y$ is a Sylow subgroup of G_{xy}. Then each orbit of P on $\Delta(x)$ has length $\geqslant q^2 r/q = qr$. In particular, if (2′) holds then P is transitive on $\Delta(x) \cap \Gamma(y)$ by (5).

LEMMA. $h \geqslant q$.

Proof. Suppose $h < q$. Certainly, P acts on $\Gamma(x) \cap \Gamma(y) - (xy - \{x, y\})$, a set of $\lambda - (h - 1) = (q\gamma - qr\delta - 1) - (h - 1)$ points (by (5)). It thus has an orbit z^P there of length $< q$. Then $|z^P| \leqslant q/p$.

Note that $\Gamma(y) \cap \Gamma(z) \cap \Delta(x)$ cannot be empty. For suppose it is. Then $y^\perp \cap z^\perp \subseteq x^\perp$, so $y^\perp \cap z^\perp = x^\perp \cap y^\perp$ (since $z \in \Gamma(y)$ implies that $|y^\perp \cap z^\perp| = |x^\perp \cap y^\perp|$). By (3), $z \in yz = xy$, which is not the case.

Thus, P_z acts on the nonempty set $\Gamma(y) \cap \Gamma(z) \cap \Delta(x)$. But each orbit of P_z on $\Delta(x)$ has length $\geqslant qr/qp^{-1} = pr > q\gamma - qr\delta - 1 = \lambda = |\Gamma(y) \cap \Gamma(z)|$ if (1′) holds. Consequently, we may assume (2′). Then $q/p = 1$, so $P = P_z$. Now P fixes $\Gamma(y) \cap \Gamma(z) \cap \Delta(x)$, and (as already noted) is transitive on $\Gamma(y) \cap \Delta(x)$. It follows that $\Gamma(y) \cap \Delta(x) = \Gamma(y) \cap \Gamma(z) \cap \Delta(x)$.

Hence, $z \in \Sigma = \{z \in \Gamma(x) \mid \Gamma(z) \cap \Delta(x) = \Gamma(y) \cap \Delta(x)\}$. Here, Σ is a set of imprimitivity of G_x on $\Gamma(x)$. Write $\sigma = |\sum|$. Then there are $q\gamma/\sigma$ different sets $\Delta(x) \cap \Gamma(y)$, $y \in \Gamma(x)$, each having size qr by (5). The number of such sets containing a given $w \in \Delta(x)$ is $(q\gamma/\sigma)(qr)/|\Delta(x)| = \gamma/\sigma$. Consequently, $p \nmid (q\gamma - \sigma) = |\Gamma(x) - \sum|$. Thus, P fixes some $z' \in \Gamma(x) - \sum$.

By (4), $h \mid (q\gamma, q\gamma - qr) = q$, so $h = 1$ and $z' \notin xy$. The preceding argument and $z' \notin \sum$ imply that $\Gamma(y) \cap \Gamma(z') \cap \Delta(x) = \phi$ and $z' \in \Delta(y)$. Recall that P is a Sylow subgroup of G_{xy}. Hence, interchanging the roles of x and y we find that P is transitive on $\Delta(y) \cap \Gamma(x)$ as well as on $\Delta(x) \cap \Gamma(y)$. Since $z' \in \Delta(y) \cap \Gamma(x)$, this is ridiculous. This contradiction proves the lemma.

Let $w \notin xy$. If $xy \cap \Gamma(w)$ contains at least two points y, y', then $x \in xy = yy' \subseteq w^{\perp}$ by (3), so $w \in x^{\perp}$. In particular, if $w \in \Delta(x)$ then $\Gamma(w)$ meets each of the $q\gamma/h \leqslant \gamma$ singular lines on x at most once, and hence exactly once since $\gamma = |\Gamma(w) \cap \Gamma(x)|$. Thus, $h = q$ and we have shown: if a point is not on a singular line L, then it is jointed by singular lines to one or all points of L. Theorems 1 and 2 thus follow from the theorem of Buekenhout–Shult [1].

Corollary 1 is now a consequence of arithmetic.

Remark 1. Our proof shows that, if (2′) is replaced by

$$\delta = 1 \quad \text{and} \quad (5) \text{ holds}, \tag{2''}$$

then singular lines have $h + 1$ points with $1 < h \mid q$.

In this case, the number of singular lines is $(1 + q\gamma + q^2 r)(q\gamma/h)/(h + 1)$, and this provides an additional restriction on the parameters q, r, δ, and h. This idea is used in the following proof.

Proof of Corollary 2. Theorem 1 applies except when $p = 2 < q$. In this case, we will imitate the proof of Theorem 1. Namely, (5) follows as before, and we need to prove $h \geqslant q$. Assume $h < q$. By (4), $h \mid q$. If $h \leqslant q/4$, we can proceed as in the lemma, with $|z^P| \leqslant q/4$ this time. Since $(4 + q)q^3 > q(q + 1)^2$, this produces a contradiction. Thus, $h = q/2$. But now the number $(1 + q(q + 1)^2 + q^4) \cdot q(q + 1)^2 h^{-1}/(h + 1)$ of singular lines is not an integer.

In view of Wagner [10] and Perin [7], this proves the corollary.

Remark 2. Primitivity in (1) could have been replaced by the condition $(1 + q\gamma) \nmid \delta$, and in (2) by $(1 + q^2 r) \nmid \gamma$.

Remark 3. Theorem 1 applies to the case of subdegrees 1, $q(t + 1)$, $q^2 t\epsilon$ with $(\epsilon, t + 1) = 1$, $q > 1$, and $t > 1$ powers of a prime p, and $t \geqslant q/p$. Here, a generalized quadrangle arises (with parameters q, t), so $\epsilon = 1$; also

$t \leqslant q^2$ by Higman [4, p. 278]. (The case $q = t = t\epsilon$ should be compared with Higman [2].)

Consequently, Theorem 1 applies to the case of the subdegrees of $PSp(4, q)$, $PGU(4, q)$, or $PGU(5, q)$, in their actions on the set of singular points *or* singular lines, except that in the case of singular points for $PGU(4, q)$ or singular lines for $PGU(5, q)$ it must be assumed that q is prime.

Remark 4. Theorems 1 and 2 can be regarded as nonexistence theorems for rank 3 groups having certain parameters. Note that each theorem implies that $\delta = 1$ or $(\gamma - 1)/r$, and greatly restricts the possibilities for γ.

For example, each of the following pairs $q\gamma$, $q^2 r\delta$ of possible subdegrees is ruled out by these theorems: 42, 32; 48, 81; 60, 500; 84, 320; and 88, 128. Each case is, however, consistent with all the numerical restrictions found in Higman [2]. (In the case 60, 500 these restrictions imply that (5) holds, so Theorem 2 applies with $q = 5$. Similarly, the possibility 50, 625 is eliminated by obtaining $h = 5$ and contradiction as in Remark 1; the possibility 88, 128 leads to the same contradiction.

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