

Some locally finite flag-transitive buildings

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1. INTRODUCTION

This paper is a continuation of Kantor [4, 6] and Ronan [10]. We will construct finite and infinite ‘geometries’ admitting flag-transitive groups and having buildings as universal covers. The first building is that of $\Omega^+(6, \mathbb{Q}_2)$; the remaining ones are new and strange. As in the aforementioned papers, our motivation is the search for finite geometries that strongly resemble buildings, and for group-theoretic situations similar to BN-pairs.

The paper is split into two independent parts. In section 2 we construct finite GABs (geometries that are almost buildings [4, 6]) with diagram \square and rank 2 residues $\text{PG}(2, 2)$. One of these is described for each odd integer $m > 1$. Each of them is covered by the $\Omega^+(6, \mathbb{Q}_2)$ building.

The GABs in Section 2 admit flag-transitive groups, and hence can be constructed as coset spaces exactly as in [6, sect. 2]. Consequently, all the relevant terminology is found in that reference (but compare 3.1 below).

In Section 3 we will construct chamber systems (Tits [13], Ronan [9]) whose diagrams are complete graphs and whose rank 2 residues are $\text{PG}(2, 2)$ or $\text{PG}(2, 8)$. This section is relatively trivial. It is of interest only as an indication of the existence of pathological locally finite flag-transitive buildings.

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2. THE GABs \square

2.1. A RATIONAL ORTHOGONAL GROUP

Equip \mathbb{Q}^7 with the usual inner product, and let u_1, \dots, u_7 be the standard orthonormal basis. Set $u_* = \sum_1^7 u_i$ and $V = u_*^\perp$.

The vectors $a_i = -7u_i + u_*$, $1 \leq i \leq 7$, span V and satisfy the conditions

$$\sum_1^7 a_i = 0, \quad (a_i, a_i) = 42, \quad \text{and} \quad (a_i, a_j) = -7 \quad \text{for } i \neq j. \quad (1)$$

Clearly, there is a subgroup $G_{(a)} \cong A_7$ of $\text{GL}(V)$ preserving $\{a_1, \dots, a_7\}$ (in fact, there is an S_7 as well).

Next, set

$$b_i = a_i + a_{i+1} + a_{i+3} \quad \text{for } 1 \leq i \leq 7, \quad (2)$$

where subscripts are taken mod 7. Then it is straightforward to check that

$$\sum_1^7 b_j = 0, \quad (b_i, b_i) = 2 \cdot 42, \quad \text{and} \quad (b_i, b_j) = 2(-7) \quad \text{for } i \neq j.$$

By (1), there is a linear transformation φ on V such that $a_i^\varphi = b_{7-i}$ for all i , and $(u^\varphi, v^\varphi) = 2(u, v)$ for all $u, v \in V$. Moreover, by (1), (2) and a simple calculation $b_i^\varphi = 2a_{7-i}$. Also, $(G_{(a)})^\varphi = G_{(b)}$ induces A_7 on $\{b_1, \dots, b_7\}$.

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There is an obvious $PG(2, 2)$ whose points are the a_i and whose lines are the triples appearing in (2). Thus, $G_{(a)(b)} = G_{(a)} \cap G_{(b)} \cong PSL(3, 2)$. Moreover, φ induces an outer automorphism of this group.

Let r be the reflection in $(a_1 - a_2)^\perp$. Then r induces the transposition (a_1, a_2) on $\{a_1, \dots, a_7\}$. Set $c_i = b'_i$. Then $b_1 = c_1, b_3 = c_3$ and $b_4 = c_4$. Set $G_{(c)} = (G_{(b)})^r$. Then $G_{(a)(b)(c)}$ is the stabilizer in $G_{(a)}$ of the partition $a_4|a_1a_2|a_3a_6|a_5a_7$. Moreover, the transformation (b_1, b_3, b_4) is in $G_{(b)(c)}$.

$b_1 = 124,$	$c_1 = 124,$	$d_1 = 2a_6,$
$b_2 = 235,$	$c_2 = 135,$	$d_2 = -a_1 + a_2 + a_5 + a_7,$
$b_3 = 346,$	$c_3 = 346,$	$d_3 = 2a_4,$
$b_4 = 457,$	$c_4 = 457,$	$d_4 = 2a_3,$
$b_5 = 561,$	$c_5 = 562,$	$d_5 = a_1 + a_2 + a_5 - a_7,$
$b_6 = 672,$	$c_6 = 671,$	$d_6 = a_1 + a_2 - a_5 + a_7,$
$b_7 = 713,$	$c_7 = 723,$	$d_7 = a_1 - a_2 + a_5 + a_7.$

(In the above table we have written $b_1 = 124$ in place of $b_1 = a_1 + a_2 + a_4$. The d_i will be defined soon.) It follows that $G_{(b)(c)} \cong (A_3 \times A_4) \cdot 2$ is the stabilizer of $\{b_1, b_3, b_4\}$ in $G_{(b)}$. Since r normalizes $G_{(a)}$, we also have $G_{(a)} \cap G_{(c)} \cong PSL(3, 2)$.

Now set $d_i = c_i^\varphi$ and $G_{(d)} = (G_{(c)})^\varphi$. Then $G_{(a)(d)} = (G_{(b)(c)})^\varphi \cong (A_3 \times A_4) \cdot 2$ and $G_{(b)(d)} = (G_{(a)(c)})^\varphi \cong PSL(3, 2) \cong G_{(c)(d)}$. Moreover, $G_{(a)(b)(c)(d)} = \langle (a_1, a_2)(a_3, a_6), (a_1, a_5)(a_2, a_7) \rangle \cong D_8$.

Set $G = \langle G_{(a)}, G_{(b)}, G_{(c)}, G_{(d)} \rangle$. In the notation of [6, sect. 2], $\Gamma = \Delta(G_{(a)}, G_{(b)}, G_{(c)}, G_{(d)})$ is a GAB in G with diagram $\begin{matrix} \bullet & \text{---} & \bullet \\ | & & | \\ \bullet & \text{---} & \bullet \end{matrix}$ where each $\bullet \text{---} \bullet$ is a $PG(2, 2)$. In fact, this is clear from the intersections we have just dealt with. (Flag-transitivity follows from [1, (3.10)].)

Note that $G = \langle G_{(a)}, (b_1, b_3, b_4) \rangle$. For, $G_{(b)} = \langle G_{(a)(b)}, (b_1, b_3, b_4) \rangle, G_{(c)} = \langle G_{(a)(c)}, G_{(b)(c)} \rangle$ and $G_{(d)} = \langle G_{(a)(d)}, G_{(b)(d)} \rangle$.

Also, $\langle \varphi, r \rangle$ induces a dihedral group of order 8 on $\{G_{(a)}, G_{(b)}, G_{(c)}, G_{(d)}\}$.

2.2. FINITE GABS

Clearly, $G_{(a)}\langle r \rangle \cong S_7$ permutes the vectors a_1, \dots, a_7 , and hence can be regarded as a group of 6×6 integral matrices with respect to the basis a_1, \dots, a_6 of V (since $a_7 = -\sum_1^6 a_i$). Also, φ produces an integral matrix, while $\varphi^{-1} = \frac{1}{2}\varphi$ has all its entries in $\mathbb{Z}[\frac{1}{2}]$. Thus, $G\langle \varphi, r \rangle = \langle G_{(a)}, \varphi, r \rangle \leq GL(6, \mathbb{Z}[\frac{1}{2}])$. If m is any odd integer then we can view all of these 6×6 matrices mod m . Whenever $D \subseteq GL(6, \mathbb{Z}[\frac{1}{2}])$ let $D^{(m)}$ be the corresponding set of matrices mod m .

The homomorphism $G \rightarrow G^{(m)}$ induces an isomorphism on $G_{(a)}$ for each $a \in \{a, b, c, d\}$, and preserves intersections among these four groups. Thus, if $m > 1$ and m is odd then $\Gamma^{(m)} = \Delta(G_{(a)}^{(m)}, G_{(b)}^{(m)}, G_{(c)}^{(m)}, G_{(d)}^{(m)})$ is a finite GAB with diagram $\begin{matrix} a & & b \\ \square & & \\ c & & d \end{matrix}$ and group $G^{(m)}$. (Once again, flag-transitivity follows from [2, (3.10)].) Clearly, $G \rightarrow G^{(m)}$ induces a cover $\Gamma \rightarrow \Gamma^{(m)}$. In 2.3 we will see that this is a universal cover.

THEOREM 1. $G^{(p)} = \Omega^\pm(6, p)$ for each prime $p \neq 2, 7$, where $G^{(p)} = \Omega^+(6, p)$ if and only if $p \equiv 1, 2$ or $4 \pmod{7}$.

PROOF. Let u_1, \dots, u_7, u_8 be the standard orthonormal basis of \mathbb{Q}^8 . Then

$$7 \left(\sum_1^8 \mathbb{Z}[\frac{1}{2}]u_i \right) + \mathbb{Z}[\frac{1}{2}]u_* = \left(\sum_1^6 \mathbb{Z}[\frac{1}{2}]a_i \right) \oplus \mathbb{Z}[\frac{1}{2}]u_* \oplus 7\mathbb{Z}[\frac{1}{2}]u_8 \tag{3}$$

since $7u_i = u_* - a_i$ for $1 \leq i \leq 7$. Moreover, the three summands on the right hand side of (3) are pairwise orthogonal.

Passing mod p we find that $GF(p)^8$ becomes an orthogonal geometry decomposed into the orthogonal sum of a 6-space and a 2-space, each of which is preserved by $G^{(p)}$. The 2-space inherits the quadratic form $7x^2 + 49y^2$, which has nontrivial zeros if and only if $(-7/p) = 1$; that is, if and only if $p \equiv 1, 2$ or $4 \pmod{7}$.

Since $(G_{(a)})' = G_{(a)}$, $G' = G$ and $(G^{(p)})' = G^{(p)}$. Thus, $G^{(p)}$ is contained in the orthogonal group specified in the theorem.

On the other hand, $(G\langle r \rangle)^{(p)}$ lies in $O^\pm(6, p)$, and $r^{(p)}$ is a reflection. Thus, $(G\langle r \rangle)^{(p)} \geq \Omega^\pm(6, p)$ by Wagner [14]. This proves that $G^{(p)} = \Omega^\pm(6, p)$, as required.

When $p = 3$, both $\Delta^{(p)}$ and the theorem were first obtained by Aschbacher and Smith [2].

2.3. THE $\Omega^+(6, \mathbb{Q}_2)$ BUILDING

Next, we will identify the GAB in (2A):

THEOREM 2. *The GAB Γ is isomorphic to the affine building for $\Omega^+(6, \mathbb{Q}_2)$.*

We will imitate the proof given in [6] of a similar result.

LEMMA 1. *$V \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is isometric to an $\Omega^+(6, \mathbb{Q}_2)$ -space.*

FIRST PROOF. By [6, (6.1)], \mathbb{Q}_2^8 is an $\Omega^+(8, \mathbb{Q}_2)$ -space. By (3), the orthogonal complement of $V \otimes_{\mathbb{Q}} \mathbb{Q}_2$ inherits the quadratic form $7x^2 + 49y^2$, and this has nontrivial zeros in \mathbb{Q}_2 .

SECOND PROOF. Let $\lambda, \bar{\lambda} = (-1 \pm \sqrt{-7})/2 \in \mathbb{Q}_2$. A straightforward calculation (using the fact that $\lambda^2 + \lambda + 2 = 0$) proves that the vectors $(\lambda, \lambda, \bar{\lambda}, \lambda, \bar{\lambda}, \bar{\lambda}, 3)$, $(3, \lambda, \lambda, \bar{\lambda}, \lambda, \bar{\lambda}, \bar{\lambda})$ and $(\bar{\lambda}, 3, \lambda, \lambda, \bar{\lambda}, \lambda, \bar{\lambda})$ span a totally singular 3-space lying in $V \otimes_{\mathbb{Q}} \mathbb{Q}_2$.

The affine building Δ for $\Omega^+(6, \mathbb{Q}_2)$ can be described as follows (Bruhat–Tits [3]). The vector space $V' = V \otimes_{\mathbb{Q}} \mathbb{Q}_2$ has a basis $e_1, e_2, e_3, f_1, f_2, f_3$ such that all inner products are 0 except for $(e_j, f_j) = (f_j, e_j) = 1$. Define the four \mathbb{Z}_2 -lattices L_i by

$$\begin{aligned} L_0 &= \langle e_1, e_2, e_3, f_1, f_2, f_3 \rangle, \\ L_1 &= \langle \tfrac{1}{2} e_1, e_2, e_3, 2f_1, f_2, f_3 \rangle, \\ L_3 &= \langle \tfrac{1}{2} e_1, \tfrac{1}{2} e_2, \tfrac{1}{2} e_3, f_1, f_2, f_3 \rangle, \\ L_{3'} &= \langle \tfrac{1}{2} e_1, \tfrac{1}{2} e_2, \tfrac{1}{2} f_3, f_1, f_2, e_3 \rangle, \end{aligned}$$

where the brackets denote generation of lattices over the ring \mathbb{Z}_2 of 2-adic integers. Let P_i be the stabilizer of L_i in $\Omega^+(6, \mathbb{Q}_2)$. Then $\Delta = \Delta(P_0, P_1, P_3, P_{3'})$. The corresponding diagram is



Note that there is an obvious dihedral group of order 8 inducing graph automorphisms on Δ .

Let $L_a = \langle a_1, \dots, a_7 \rangle$, and define L_b, L_c and L_d similarly.

LEMMA 2. *We may assume that L_a is a scalar multiple of L_0 .*

PROOF. $G_{(a)}$ fixes some point of the realization of Δ (Bruhat–Tits [3, pp. 64–65]). Since $G_{(a)} \cong A_7$ it fixes a vertex. Thus, we may assume that $G_{(a)} < P_0$. Choose $\kappa \in \mathbb{Q}_2$ so that $\kappa a_1 \in L_0 - 2L_0$. Then $L_0 \supseteq \kappa L_a + 2L_0 \supset 2L_0$. By the irreducibility of $G_{(a)}$, $L_0 = \kappa L_a + 2L_0$.

Since $L_0/L_0 \cap \kappa L_a \subseteq 2(L_0/L_0 \cap \kappa L_a)$, $L_0 \subseteq \kappa L_a$ by Nakayama's Lemma [7, p. 242]. Thus, $L_0 = \kappa L_a$.

LEMMA 3. *We may assume that L_b, L_c and $L_a \cap (\frac{1}{2}L_d)$ are scalar multiples of L_3, L_3 and $L_0 \cap L_1$, respectively.*

PROOF. By definition, $L_a \supset L_b$. Applying φ , we find that $L_b \supset 2L_a$. Now apply r in order to deduce that $L_a \supset L_c \supset 2L_a$. Also, $L_a \cap (\frac{1}{2}L_d) \supseteq 2L_a$. (For, $L_a \cap (\frac{1}{2}L_d)$ contains $a_6, a_4, a_3, a_1 + a_2, a_1 - a_2, a_1 + a_5$ and $a_1 - a_5$.)

Thus, the group $S = G_{(a)(b)(c)(d)}$ fixes the following three subspaces of $L_a/2L_a$:

$$L_b/2L_a = \langle b_1 + 2L_a, b_2 + 2L_a, b_3 + 2L_a \rangle$$

$$L_c/2L_a = \langle c_1 + 2L_a, c_2 + 2L_a, c_3 + 2L_a \rangle$$

$$L_a \cap (\frac{1}{2}L_d)/2L_a = \langle a_6 + 2L_a, a_4 + 2L_a, a_3 + 2L_a, a_1 + a_2 + 2L_a, a_1 + a_5 + 2L_a \rangle.$$

Also, $L_a/2L_a$ inherits the structure of an $\Omega^+(6, 2)$ -space (the quadratic form being $v \mapsto (v, v)/2 \pmod{2}$). The subspaces $L_b/2L_a, L_c/2L_a$ and $(L_a \cap (\frac{1}{2}L_d)/2L_a)^\perp = \langle d_2 + 2L_a \rangle$ are totally singular and pairwise incident (since $b_1 = c_1$ and $b_3 = c_3 = -d_2 - 2a_1$), and hence form a flag of the $\Omega^+(6, 2)$ -space $L_a/2L_a$. This is the only such flag fixed by S .

If $L_0 = \kappa L_a$ then S fixes a flag of $\kappa L_a/2\kappa L_a = L_0/2L_0$. Choose notation first so that this flag is $2L_3/2L_0, 2L_3'/2L_0, (L_0 \cap L_1/2L_0)^\perp$ and then so that $2L_3 = \kappa L_b, 2L_3' = \kappa L_c$, and $L_0 \cap L_1 = \kappa(L_a \cap (\frac{1}{2}L_d))$.

PROOF OF THEOREM 2. By Lemmas 2 and 3, $G_{(a)} < P_0, G_{(b)} < P_3, G_{(c)} < P_3'$ and $G_{(a)(d)} < P_0 \cap P_1$. Set $r' = r^\varphi$. Then r' interchanges L_a and $\frac{1}{2}L_d$, and hence normalizes the stabilizers $P_0 \cap P_1$ of $L_a \cap (\frac{1}{2}L_d)$. Thus, $G_{(d)} = (G_{(a)})^{r'} < P_0'$ where $P_0 \cap P_1 < P_0'$. Since $G_{(d)} \not\leq P_0$, it follows that $P_0' = P_1$. This proves that $G_{(a)} < P_0, G_{(b)} < P_3, G_{(c)} < P_3'$ and $G_{(d)} < P_1$.

It follows that G induces a flag-transitive group on the residue of each of the vertices P_i of Δ . Since Δ is connected, it follows that G is flag-transitive on Δ .

Now define $\Gamma \rightarrow \Delta$ via $G_{(a)g} \mapsto P_0g$ and so on (where $g \in G$). This is a cover. Since Δ is simply connected (Tits [13]), it follows that $\Gamma \cong \Delta$.

REMARK. By considering their discriminants it is easy to show that $L_a = L_0, L_b = L_3, L_c = L_3'$ and $\frac{1}{2}L_d = L_1$.

2.4. THE GROUP G

Let f be the quadratic form $42\sum_1^6 x_i^2 - 14\sum_{1 \leq i < j \leq 6} x_i x_j$ obtained from (1) by using the basis a_1, \dots, a_6 of V . Let $\Omega(\mathbb{Z}[\frac{1}{2}], f)$ be the commutator subgroup of the corresponding orthogonal group over $\mathbb{Z}[\frac{1}{2}]$.

THEOREM 3. $G = \Omega(\mathbb{Z}[\frac{1}{2}], f)$.

PROOF. By 2.2, $G \leq \text{GL}(6, \mathbb{Z}[\frac{1}{2}])$. Also, $G = G'$ and G preserves the form f obtained by restricting to V the usual form on \mathbb{Q}^7 . Thus, $G \leq \Omega = \Omega(\mathbb{Z}[\frac{1}{2}], f)$.

By Theorem 1 and Lemma 1, Ω acts flag-transitively on Δ . Then $\Omega = (\Omega \cap P_0)G$. We will show that $\Omega \cap P_0 \leq G$.

Let $g \in \Omega \cap P_0$. The matrix (x_{ij}) of g with respect to the basis a_1, \dots, a_6 must have all entries x_{ij} in $\mathbb{Z}[\frac{1}{2}]$. On the other hand, Lemma 2 implies that $(L_a)^g = L_a$, so that $x_{ij} \in \mathbb{Z}_2$. Thus, $x_{ij} \in \mathbb{Z}$. Since g preserves f , there are only finitely many possibilities for (x_{ij}) .

The centralizer C of $L_a/2L_a$ in $\Omega \cap P_0$ is a finite subgroup normalized by $G_{(a)}$, and $C \cap G_{(a)} = 1$. Then C consists of scalars. But $-1 \notin \Omega$ (since $-1 \notin \Omega^-(6, 17) = \Omega^{(17)}$). Thus, $C = 1$ and $\Omega \cap P_0$ is isomorphic to a subgroup of $\Omega^+(6, 2)$ containing $G_{(a)} \cong A_7$. Since $\Omega^+(6, 2) \cong A_8$ is not isomorphic to any subgroup of $GL(6, \mathbb{Q})$, it follows that $\Omega \cap P_0 = G_{(a)} < G$, as required.

2.5. FURTHER PROPERTIES

We conclude this section with several remarks concerning G and Γ .

2.5.1. Assume that $G^{(p)} = \Omega^\pm(6, p)$ with $p \equiv \pm 1 \pmod{4}$. Then $-1 \in G^{(p)}$, while $-1 \notin G_{(x)}^{(p)}$ for each x . Let ‘bar’ denote the homomorphism $G^{(p)} \rightarrow G^{(p)}/\langle -1 \rangle$. Then $\bar{\Gamma}^{(p)} = \Delta(\bar{G}_{(a)}^{(p)}, \bar{G}_{(b)}^{(p)}, \bar{G}_{(c)}^{(p)}, \bar{G}_{(d)}^{(p)})$ is a finite GAB with group $\bar{G}^{(p)}$, and there is an obvious cover $\Gamma^{(p)} \rightarrow \bar{\Gamma}^{(p)}$. This is a 2-fold cover. For, -1 acts nontrivially on $\Gamma^{(p)}$ since -1 does not fix $G_{(a)}^{(p)}$.

Note that this is quite different from the situation in [6], where -1 always acted trivially.

2.5.2. By construction, $\langle \varphi, r \rangle^{(m)}$ induces a D_8 of graph automorphisms of $\Gamma^{(m)}$.

If $p \neq 2$ is a prime and $(2/p) = 1$ then $\langle \varphi, r \rangle^{(p)} < G^{(p)}$: the graph automorphisms are induced by inner automorphisms. In particular, for such primes p all $G_{(x)}^{(p)}$ are conjugate in $G^{(p)}$.

2.5.3. In the case $p = 7$ not covered by Theorem 1, $G^{(7)} = 7^5 \Omega(5, 7)$. For, if $a'_i = a_i/7$ then G acts on

$$\sum_1^6 \mathbb{Z}[\frac{1}{2}]a'_i = \sum_2^6 \mathbb{Z}[\frac{1}{2}](a'_i - a'_i) + \mathbb{Z}[\frac{1}{2}]a'_1,$$

while $a'_1 - a'_7 = -\sum_2^6(a'_i - a'_i) + 7a'_1$ and $(a'_1 - a'_i, a'_i - a'_j)$ is 1 if $i \neq j$ and 2 if $i = j$. Passing mod 7 we obtain a 6-space V_6 over $GF(7)$ and a hyperplane V_5 upon which $G^{(7)}$ acts. Moreover, V_5 inherits a nontrivial G -invariant inner product (although V_6 does not). Using $(G\langle r \rangle)^{(7)}$ as in the proof of Theorem 1, we find that G induces $\Omega(5, 7)$ on V_5 . On the other hand, $(G_{(a)})^{(7)}$ does not fix any 1-space of V_6 . From this it follows that $G^{(7)}$ cannot act faithfully on V_5 , and hence that $G^{(7)}$ is as claimed.

Moreover, the homomorphism $G^{(7)} \rightarrow \Omega(5, 7)$ induces a 7⁵-fold cover from $\Gamma^{(7)}$ onto a GAB with diagram \square and group $\Omega(5, 7)$.

2.5.4. The situation in 2.3 closely resembles that of [6]. This relationship can be made more precise, as follows. Let u_1, \dots, u_8 be as in the proof of Theorem 1. Set $v_* = u_* + u_8 = \sum_1^8 u_i$. Regard $G_{(a)}\langle r \rangle$ as a group of isometries of \mathbb{Q}^8 permuting $\{u_1, \dots, u_8\}$ and fixing u_8 . Extend φ to \mathbb{Q}^8 by letting $u_8^\varphi = \frac{1}{2}v_*$ and $v_*^\varphi = 4u_8$. Then $(u^\varphi, v^\varphi) = 2(u, v)$ for all $u, v \in \mathbb{Q}^8$, so that $G\langle r, \varphi \rangle$ projectively preserves the form $\sum_1^8 x_i^2$. Call this form f_8 , and set $G_8 = \Omega(\mathbb{Z}[\frac{1}{2}], f_8)$. Then $G = (G_8)_{u_8, v_*}$. Moreover, if r_8 and r_* are the reflections in u_8 and v_* , then both lie in G_8 while $G = C_{G_8}(\langle r_8, r_* \rangle)$. Finally, the complex Γ in 2.2 can be identified with the set of fixed points of $\langle r_8, r_* \rangle$ on the complex Δ_8 occurring in [6, sect. 5].

2.5.5. Three vectors were introduced in the second proof of Lemma 1; call them v_1, v_2, v_3 . Set $v_4 = u_* + \sqrt{-7} u_8$. Let \bar{v}_i be defined by replacing $\sqrt{-7}$ by $-\sqrt{-7}$. Then v_1, v_2, v_3, v_4 and $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ span complementary totally singular 4-spaces of \mathbb{Q}_2^8 .

2.5.6. As in [6, (10.3)], it can be shown that any finite flag-transitive GAB with diagram \square arises as an image of Γ in such a way that the commutator subgroup of the given group

is a homomorphic image of G . More generally, a similar results holds for chamber-transitive SCABs (cf. 3.1) with the above diagram.

3. COMPLETE GRAPH DIAGRAMS

The present section is based on a generalization of GABs.

3.1. CHAMBER SYSTEMS AND COSETS

A *chamber system* of rank n consists of a set \mathcal{C} of objects called *chambers*, together with a family of partitions \mathcal{E}_i of \mathcal{C} , where i ranges through a set I of size n . If $J \subseteq I$ then \mathcal{E}_J denotes the join of the partitions $\mathcal{E}_j, j \in J$. The chamber system is *connected* if $\mathcal{E}_I = \mathcal{C}$. A member of \mathcal{E}_i is called an *i -edge*. A *vertex* is a member of $\mathcal{E}_{I-\{i\}}$ for some $i \in I$.

If $x \in \mathcal{E}_J, J \subseteq I$, then the *residue* $\text{Res}(x)$ is the chamber system consisting of the set of chambers in x , together with the intersections of this set with all $\mathcal{E}_j, j \in J$. Clearly, $\text{Res}(x)$ has rank $|J|$. A chamber system is *residually connected* if $\text{Res}(x)$ is connected whenever $x \in \mathcal{E}_J$ for some J of size $\leq n - 2$.

DEFINITION. A chamber system is a *SCAB* (chamber system that is almost a building) if it is residually connected of rank $n \geq 3$ and if, for each 2-set $\{i, j\} \subseteq I$, there is an integer n_{ij} such that $\text{Res}(x)$ is a generalized n_{ij} -gon for each $x \in \mathcal{E}_{\{i,j\}}$. The *diagram* of the SCAB is defined as follows: it has node set I ; the distinct nodes i and j are joined by $n_{ij} - 2$ edges if $n_{ij} \in \{2, 3, 4\}$, by $\frac{1}{2}n_{ij}$ edges if $n_{ij} = 6$ or 8 , and by an edge labeled n_{ij} in all other cases.

In this terminology, a GAB can be regarded as a SCAB in which every flag has a nonempty intersection. (Here, a *flag* is a set of vertices of the SCAB any two of which have a nonempty intersection. Compare Tits [11, p. 3] and [13].)

We will only be interested in connected chamber systems that admit an automorphism group G transitive on \mathcal{C} . In this situation, fix $C \in \mathcal{C}$, let B be its stabilizer in G , and let E_i be the stabilizer of the i -edge containing C . Then the chamber system is isomorphic to the chamber system $(G, B, E_i)_{i \in I}$ defined as follows: chambers are cosets of B ; and each i -edge is a coset of E_i , regarded as a set of cosets of B . Conversely if G is a group generated by a family $E_i, i \in I$, of subgroups, and $B \leq \cap \{E_i | i \in I\}$, then the above definition produces a connected chamber system with a chamber-transitive automorphism group. If $E_j = \langle E_j | j \in J \rangle$ then \mathcal{E}_J can be identified with the set of cosets of E_J .

A theorem of Tits [13] states that, if $\mathcal{C}, \mathcal{E}_i (i \in I)$ is a SCAB, and if every rank 3 residue having a spherical diagram is a building, then there is a universal 2-covering SCAB $\tilde{\mathcal{C}}, \tilde{\mathcal{E}}_i (i \in I)$ that is a building. Thus, there is a surjection $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ such that, whenever $J \subseteq I$ and $|J| \leq 2$, every member of $\tilde{\mathcal{E}}_J$ is mapped bijectively onto a member of \mathcal{E}_J . Consequently, rank 2 residues are mapped isomorphically. If the original SCAB has the form $(G, B, E_i)_I$, then its universal 2-cover also has a chamber-transitive automorphism group, and hence has the form $(\tilde{G}, \tilde{B}, \tilde{E}_i)_I$, where $G = \tilde{G}/N$ for the group N of covering transformations (compare Ronan [9]). Moreover, $B = \tilde{B}N/N$ and $E_i = \tilde{E}_iN/N$.

In the remainder of this section we will have $B = 1$, so that G will be regular on chambers.

3.2. FROBENIUS GROUPS

If a SCAB has as diagram the complete graph K_n on $n = |I|$ vertices, and if each edge corresponds to $\text{PG}(2, q)$, we will say that the diagram is $K_n/\text{PG}(2, q)$.

Let $q = 2$ or 8 .

REMARK. If G is a group generated by a finite set $\{E_i | i \in I\}$ of $n \geq 3$ subgroups of order $q + 1$, any two of which generate a Frobenius group of order $(q^2 + q + 1)(q + 1)$, then $(G, 1, E_i)_I$ is a SCAB with diagram $K_n/PG(2, q)$.

PROOF. If $j \neq k$ then $(E_{\{j,k\}}, 1, E_i)_{\{j,k\}}$ is $PG(2, q)$, since the Frobenius group $E_{\{j,k\}}$ is sharply flag-transitive on that plane. (NB: This flag-transitivity holds only for $q = 2$ or 8 .)

3.3. EXAMPLES WITH DIAGRAM $K_n/PG(2, q)$

As in 3.2, let $q = 2$ or 8 . It is easy to give many examples of SCABs using 3.2. The following examples are not GABs. Nevertheless, each has as universal cover a flag-transitive building [13].

EXAMPLE 1. Let $F = GF(q^2 + q + 1)$ and $m \geq 1$, and let G be the semidirect product of F^m with a cyclic group $\langle t \rangle$ of order $q + 1$, where t induces a scalar transformation on F^m . Then G produces many SCABs with diagrams $K_n/PG(2, q)$, $n \leq (q^2 + q + 1)^m$, as follows. Let $v_1 = 0, \dots, v_n$ be any set of elements generating F^m . Then $\{\langle v_i t \rangle | 1 \leq i \leq n\}$ satisfies the requirements of 3.2. For, if $i \neq j$ then $\langle v_i t, v_j t \rangle = \langle v_i v_j^{-1}, v_i t \rangle$ where $(v_i v_j^{-1})^{v_i t} = v_i v_j^{-s} = (v_i v_j^{-1})^s$ for some integer s satisfying $0 < s < q^2 + q + 1$.

Of course, each of these SCABs is a residue of a certain one having diagram $K_N/PG(2, q)$, where $N = (q^2 + q + 1)^m$. In fact, if we allowed infinite index sets I then all of these for all m would be residues of a SCAB having as diagram a countable complete graph.

It is easy to check that these SCABs are never GABs.

Note that if $m = 2 = n - 1$ then the above SCAB has diagram $K_3/PG(2, q)$.

EXAMPLE 2. The group G of all transformations $x \rightarrow ax^q + b$ over $F = GF(q^3)$, $q = 2$ or 8 , $a^{q^2+q+1} = 1, b \in F, \sigma \in \text{Aut } F$, produces a SCAB with diagram $K_3/PG(2, q)$, as follows. Note that F contains an element α satisfying $\alpha^{q+1} = \alpha + 1$. Then $\alpha^{q^2+q+1} = \alpha(\alpha^{q+1})^q = \alpha(\alpha^q + 1) = 1$. The three groups $G_{01}, G_{0\alpha}$ and $G_{1\alpha}$ have order $q + 1$. For, this is clear in the case of G_{01} , while $G_{0\alpha}$ and $G_{1\alpha}$ are obtained by conjugating by $x \rightarrow \alpha x$ and $x \rightarrow (\alpha + 1)x + \alpha$, respectively. Then 3.2 applies to $\{G_{01}, G_{0\alpha}, G_{1\alpha}\}$.

This example can be generalized, as follows.

EXAMPLE 3. Let $q = 2$ or 8 , let $m \geq 1$, and let $G^{(m)}$ be the group obtained from the direct product of m copies of the group G in Example 2 by identifying all the m versions of $x \rightarrow x^2$; thus, $|G| = q^{3m}(q^2 + q + 1)^m(q + 1)$. Then $G^{(m)}$ produces a SCAB with diagram $K_{2m+1}/PG(2, q)$. For, let $t \in G^{(m)}$ have order $q + 1$. In each copy of G pick elements a_i, b_i of order $q + 1$ such that Example 2 applies to $\{t, a_i t, b_i t\}$ for $1 \leq i \leq m$. Then 3.2 applies to $\{\langle t \rangle, \langle a_i t \rangle, \langle b_i t \rangle | 1 \leq i \leq m\}$. For, $\langle a_i t, b_i t \rangle = \langle a_i b_i^{-1}, a_i t \rangle$. If $i = j$ our choice of a_i and b_j yields that this is a Frobenius group. If $i \neq j$ then $|a_i b_j^{-1}| = q^2 + q + 1$ and $(a_i b_j^{-1})^{a_i t} = a_i^2 b_j^{-2} = (a_i b_j^{-1})^2$, as required.

Note that the resulting SCAB is a covering of one of the SCABs in Example 1.

EXAMPLE 4. $G = P\Gamma L(2, q^3)$, $q = 2$ or 8 , produces a SCAB with $(q^3 + 1)(q - 1)$ vertices and diagram $K_4/PG(2, q)$. For, if α is as in Example 2 then 3.2 can be applied to $\{G_{x01}, G_{x0\alpha}, G_{x1\alpha}, G_{01\alpha}\}$ (where the subscripts refer to stabilizers when G is regarded as acting on the projective line $\{\infty\} \cup GF(q^3)$). That the first three of these behave as desired was shown in Example 2. Any element of $PSL(2, q^3)$ interchanging ∞ and 0 and sending 1 to α must be an involution, and hence leaves invariant our set of four groups. Hence, any three of the groups behave as desired.

Of course, all rank 3 residues are as in Example 2. As in Examples 2 and 3, we can generalize this example as follows.

EXAMPLE 5. Let G be the semidirect product of the direct product of $n \geq 1$ copies of $\text{PSL}(2, q^3)$, $q = 2$ or 8 , by a cyclic group $\langle t \rangle$ of order $q + 1$, where t induces the field automorphism $x \rightarrow x^2$ on each factor. Then G produces a SCAB with diagram $K_{3n+1}/\text{PG}(2, q)$. For, let a_i, b_i, c_i be elements of the i th factor such that $\{\langle t \rangle, \langle a_i t \rangle, \langle b_i t \rangle, \langle c_i t \rangle\}$ is as in Example 4. Then the union of these n sets behaves as in 3.2. Namely, using Example 2 we see that $a'_i = a_i^2, b'_i = b_i^2$ and $c'_i = c_i^2$. Thus, we can proceed exactly as in Example 3.

Finally, Examples 1, 3 and 5 can be merged in a similar manner:

EXAMPLE 6. Consider the group $(A \times B \times C)\langle t \rangle$ where A, B, C and t are as follows: $A = \text{GF}(q^2 + q + 1)^k$ and $a^t = 2a$ for all $a \in A$; $B\langle t \rangle$ is the group in Example 3; and $C\langle t \rangle$ is the group in Example 5. Then $(A \times B \times C)\langle t \rangle$ produces many SCABs with diagram $K_N/\text{GF}(q)$, the largest N being $N = (q^2 + q + 1)^k + 2m + 3n$.

PROBLEM. When can two buildings constructed as in the above examples (via [13]) be isomorphic?

Note that Example 6 shows that any two such buildings arise as residues of one of these buildings.

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