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## Flag-Transitive Planes

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The fundamental theorem of Ostrom and Wagner [6] states that a finite projective plane admitting a 2-transitive collineation group must be desarguesian. Various weaker hypotheses have been considered by many authors (e.g., [1-10]). Let $\pi$ be a finite projective plane having a point-transitive collineation group F. Wagner [10] showed that $\pi$ is desarguesian if F contains a nontrivial perspectivity. Higman and McLaughlin [3] introduced the study of "flag-transitive projective planes," where F is transitive on the flags (i.e., nonincident point-line pairs) of $\pi$. Results in [1, p. 212; 2; 3] state that such a plane must be desarguesian if its order is not a fourth power and the number of points is not a prime.

Ott [7,8] showed that a flag-transitive plane has prime power order, unless F is sharply flag-transitive. His proofs involved the use of group theory. Using the classification of finite simple groups, it is possible to go much further:

THEOREM A If $\pi$ is a finite projective plane of order n having v points, and if F is a flag-transitive collineation group of $\pi$, then either (i) $\pi$ is desarguesian and F contains PSL(3,n), or (ii) F is a Frobenius group of odd order $v(n + 1)$, and v is a prime.

By [3], the group F is necessarily primitive in its action on the set of points. This fact, together with the observation that v is odd, are what turn out to be relevant in the proof of the theorem. In fact, primitivity alone is all that is needed.

THEOREM B If $\pi$ is a finite projective plane of order n having v points, and if F is a point-primitive collineation group of $\pi$, then either (i) $\pi$ is desarguesian and F contains PSL(3,n), or (ii) F is a Frobenius group of odd order dividing v(n + 1) or vn, and v is a prime.

Consequently, our proof of Theorem A does not use any of the abovementioned results, except for Wagner's theorem [10] and primitivity. In fact, assuming flag-transitivity would only slightly shorten the proof. It should be noted that part (ii) of the conclusions is concerned with difference sets, and this is a situation very different from that in the group theoretic approach leading to conclusion (i).

The proof of Theorem B involves very detailed knowledge of primitive permutation groups of odd degree, which is obtained in [4] using the classification of finite simple groups. However, there are so many different types of permutation groups of this sort that consideration of all of them in Theorem B is neither straightforward nor brief. The tedious and somewhat ludicrous details are given in [4].

OPEN PROBLEMS 1. Show that a finite affine plane is a translation plane if it admits a point-primitive collineation group F. If the plane has even order then it is easy to show that F contains a nontrivial translation (cf. [6,9]), and the required result follows immediately. In the odd order case the group theoretic result in [4] applies, and should produce the desired result without too much difficulty.

2. Classify all designs with $\lambda = 1$ and v odd admitting a flag-transitive automorphism group that has no regular normal subgroup. This generalization of Theorem A is not as outrageous as it may sound. Once again the group theoretic result in [4] provides a significant amount of information. What is needed is a new way to use flag-transitivity more efficiently than in [2], where Baer involutions were available.

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