Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 6, No. 2 (2003) 1–15

SINGULAR SPACETIME ITÔ'S INTEGRAL AND A CLASS OF SINGULAR INTERACTING BRANCHING PARTICLE SYSTEMS

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Received 7 June 2002
Communicated by D. A. Dawson

In Wang,[8] a class of interacting measure-valued branching diffusions $\{\mu_t^\varepsilon, t \geq 0\}$ with singular coefficient were constructed and characterized as a unique solution to $\mathcal{L}^\varepsilon$-martingale problem by a limiting duality method since in this case the dual process does not exist. In this paper, we prove that for any $\varepsilon \neq 0$ the superprocess with singular motion coefficient is just the super-Brownian motion. The singular motion coefficient is handled as a sequential limit motivated by Ref. 1. Thus, the limiting superprocess is investigated and identified as the motion coefficient converges to a singular function. The representation of the singular spacetime Itô's integral is derived.

Keywords: Superprocess; interaction; singularity.

AMS Subject Classification (2000): 60K35, 60J80

1. Introduction

In Wang,[10] a class of interacting branching particle systems has been studied and their limiting superprocesses have been constructed and characterized. In this paper, we consider that if the coefficient of the spacetime Itô's integral in the interacting branching particle systems considered in Ref. 10 converges to a singular function, how to define the singular spacetime Itô's integral and what are the singular interacting branching particle systems and the corresponding superprocess? Here we need to give a precise definition of a singular function.

Definition 1.1. A function $f(x)$ defined on $\mathbb{R}$ is called a *singular function with singular point* $y \in \mathbb{R}$ if there exists a convergent sequence $\{f_n\}$ which converge to f pointwise and $|f(y)| = \infty$.

For fixed natural integer k, $m \geq 1$, let $C^k(\mathbb{R}^m)$ be the set of functions on $\mathbb{R}^m$ having continuous derivatives of order $\leq k$ and $C^k_\partial(\mathbb{R}^m)$ be the set of functions in $C^k(\mathbb{R}^m)$ which together with their derivatives up to the order k can be extended continuously to $\bar{\mathbb{R}}^m := \mathbb{R}^m \cup \{\partial\}$, the one point compactification of $\mathbb{R}^m$. Let $C_0^k(\mathbb{R}^m)$ be the subset of $C_\partial^k(\mathbb{R}^m)$ of functions that together with their derivatives up to the order k vanishing at infinity. The convergent sequence is defined as follows.

Definition 1.2. A sequence of functions $\{f_n \in C_\partial^k(\mathbb{R}) : n \geq 1\}$ with fixed $k \geq 1$ is called a *convergent sequence* if there exists a point $y \in \mathbb{R}$ such that the following conditions are satisfied:

(1) For each $n \geq 1$,

$$\int_{\mathbb{R}} f_n^2(x)dx < \infty \quad \text{and} \quad G := \sup_n \int_{\mathbb{R}} |f_n(x)|dx < \infty\,. \tag{1.1}$$

(2) For any $\delta_1 > 0$ and $\delta_2 > 0$, there exists an $N > 0$ such that for any $n \geq N$ we have

$$\sup_{|x-y|>\delta_1} |f_n(x)| < \delta_2\,. \tag{1.2}$$

(3) The following limits exist and

$$\lim_{n\to\infty} |f_n(y)| = \infty\,, \quad \rho(y) := \lim_{n\to\infty} \rho_n(y) < \infty\,, \tag{1.3}$$

where

$$\rho_n(y) := \int_{\mathbb{R}} f_n(y-x)f_n(x)dx\,. \tag{1.4}$$

Roughly speaking, a singular function is a function which assumes infinite value and is defined by the sequential limit of a convergent sequence of continuously differentiable functions. To consider the above questions, we introduce the interacting branching particle system as follows. For any natural number $\hat{n}$, which is served as a control parameter of the sequence of the branching particle systems. we consider a system of particles (initially, there are $m_0^{\hat{n}}$ particles) which move, die and produce offspring in a random medium on $\mathbb{R}$. Let $E := \mathbf{M}_F(\mathbb{R})$ be the Polish space of all finite Radon measures on $\mathbb{R}$ with the weak topology defined by

$$\mu^n \Rightarrow \mu \quad \text{if and only if} \quad \langle f, \mu^n\rangle \to \langle f, \mu\rangle \forall\ f \in C_b(\mathbb{R})\,.$$

Let $(T_t^m)_{t\geq 0}$ denote the transition semigroup of the m-dimensional generalized Brownian motion with a constant covariance $\rho(0) + \varepsilon^2$ and let $(S_t^{m,k})_{t\geq 0}$ denote the transition semigroup generated by the operator $G_\varepsilon^{m,k}$ defined by (2.29). Same as that handled in Wang,[10] we can extend related functions and operators to $\bar{\mathbb{R}}^m$ whenever it is necessary. For example, $(T_t^m)_{t\geq 0}$, $(S_t^{m,k})_{t\geq 0}$, and $G_\varepsilon^{m,k}$ can be extended to $(\hat{T}_t^m)_{t\geq 0}$, $(\hat{S}_t^{m,k})_{t\geq 0}$, and $\hat{G}_\varepsilon^{m,k}$ on $\bar{\mathbb{R}}^m$ with ∂ as a trap, respectively. However, according to Theorem 4.1 of Ref. 3, for each $\mu \in E$ all superprocesses discussed in this paper live in E. Similar to the situation handled by Konno–Shiga,[5] we will give definitions and discussions on $\mathbb{R}$. When extensions are necessary, we briefly point out.

The diffusive part of such a system has the form

$$dx_i^{\hat{n},k}(t) = \int_{\mathbb{R}} g_k(y - x_i^{\hat{n},k}(t))W(dy, dt) + \varepsilon dB_t^i\,, \tag{1.5}$$

where $W(\cdot,\cdot)$ is a Brownian sheet (time-space white noise or cylindrical Brownian motion, the reader is referred to Walsh[7] or Example 7.1.2 in Ref. 2 for more details), $\{B_t^i\}$ are independent one-dimensional Brownian motions which are independent of W, ε is a real constant, and $\{g_k : k \geq 1\}$ is a sequence of functions from $C_\partial^2(\mathbb{R})$ such that for each $k \geq 1$,

$$\int_{\mathbb{R}} |g_k(x)|dx < \infty\,, \quad \int_{\mathbb{R}} g_k^2(x)dx < \infty\,. \tag{1.6}$$

The quadratic variational process for the finite system defined by (1.5) is

$$\langle x_i^{\hat{n},k}(t), x_j^{\hat{n},k}(t)\rangle = \int_0^t \rho_k(x_i^{\hat{n},k}(s) - x_j^{\hat{n},k}(s))ds + \varepsilon^2 t\delta_{\{i=j\}}\,, \tag{1.7}$$

where we set $\delta_{\{i=j\}} = 1$ or 0 according as $i = j$ or $i \neq j$ and

$$\rho_k(z) = \int_{\mathbb{R}} g_k(z - y)g_k(y)dy\,. \tag{1.8}$$

Here $x_i^{\hat{n},k(t)}$ is the location of the ith particle. We assume that each particle has mass $1/\theta^{\hat{n}}$ and branches at rate $\gamma\theta^{\hat{n}}$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants and the initial empirical measure $\mu_0^{\hat{n}}(\cdot) := \frac{1}{\theta^{\hat{n}}}\sum_{i=1}^{m_0^{\hat{n}}} \delta_{x_i^{\hat{n}}(0)}(\cdot)$ weakly converges to a finite measure μ_0 as $\hat{n} \to \infty$. As for the branching part, we assume that when a particle dies, it produces j particles with probability p_j; $j = 0, 1, 2, \ldots$. The offspring distribution is assumed to satisfy:

$$p_1 = 0\,, \quad \sum_{k=0}^{\infty} jp_j = 1 \quad \text{and} \quad m_2 := \sum_{j=0}^{\infty} j^2 p_j < \infty\,. \tag{1.9}$$

The second condition indicates that we are solely interested in the critical case. After branching, the resulting set of particles evolve in the same way as the parent and they start off from the parent particle's branching site. Let $m_t^{\hat{n}}$ denote the total number of particles at time t. Denote the empirical measure process by

$$\mu_t^{\hat{n},k}(\cdot) := \frac{1}{\theta^{\hat{n}}}\sum_{i=1}^{m_t^{\hat{n}}} \delta_{x_i^{\hat{n},k}(t)}(\cdot)\,. \tag{1.10}$$

Then, for each fixed $k \geq 1$, $\mu_t^{\hat{n},k}$ or its subsequence has a unique weak limit which is characterized as unique solution to $(\mathcal{L}_k^\varepsilon, \delta_{\mu_0})$-martingale problem (MP), where μ_0 is a finite measure on $\mathbb{R}$ and the generator is defined as follows:

$$\mathcal{L}_k^\varepsilon F(\mu) := \mathcal{A}_k^\varepsilon F(\mu) + \mathcal{B}F(\mu)\,, \tag{1.11}$$

where

$$\mathcal{B}F(\mu) := \beta \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2} \mu(dx) \,, \tag{1.12}$$

$$\begin{aligned} \mathcal{A}_k^{\varepsilon} F(\mu) := & \frac{1}{2} \int_{\mathbb{R}} \rho_{\varepsilon,k} \left(\frac{d^2}{dx^2}\right) \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_k(x-y) \left(\frac{d}{dx}\right) \left(\frac{d}{dy}\right) \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \mu(dx)\mu(dy) \end{aligned} \tag{1.13}$$

for any bounded continuous function $F(\mu)$ that belongs to the domain $\mathcal{D}(\mathcal{L}_k^{\varepsilon})$ of operator $\mathcal{L}_k^{\varepsilon}$, where the variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{h\downarrow 0} \frac{F(\mu + h\delta_x) - F(\mu)}{h} \,, \tag{1.14}$$

$$\rho_{\varepsilon,k} := \rho_k(0) + \varepsilon^2 \,, \quad \varepsilon \in \mathbb{R} \,, \tag{1.15}$$

$\beta \equiv \gamma(m_2 - 1)/2$ is a non-negative constant, m_2 is the finite second moment of the offspring distribution for the branching mechanism (refer to Ref. 10 for more details).

In this paper, the following problem is considered. We know that the Dirac delta function is defined as follows:

$$\dot{\delta}_x(y) = \begin{cases} \infty & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \tag{1.16}$$

and for any test function ϕ,

$$\int_{\mathbb{R}} \phi(y)\dot{\delta}_x(y)dy = \phi(x) \,, \tag{1.17}$$

or in an equivalent sequential definition,

$$\dot{\delta}_x(y) = \lim_{k\to\infty} p_k(x,y) \,, \tag{1.18}$$

where

$$p_k(x,y) = \frac{1}{\sqrt{2\pi(1/k)}} \exp\left\{-\frac{(x-y)^2}{2(1/k)}\right\} \tag{1.19}$$

is the heat kernel and

$$\int_{\mathbb{R}} \phi(y)\dot{\delta}_x(y)dy := \lim_{k\to\infty} \int_{\mathbb{R}} \phi(y)p_k(x,y)dy = \phi(x) \,. \tag{1.20}$$

The sequential definition of the Dirac delta function gave us the motivation for the following question. In our particle system, if we assume that as $k \to \infty$, $g_k(\cdot)$ converges to a function which is very similar to a Dirac delta function. Then, what is the limiting branching particle system as $k \to \infty$ and what is the corresponding limiting superprocess? To answer this question, we are automatically involved in the problem on how to define a singular spacetime Itô's integral. Let us start by analyzing an example to find what are necessary conditions for the functions $\{g_k(\cdot)\}$ in order to define the singular branching particle systems and corresponding limiting superprocess. For the sake to find the problems easily, in the following example we only consider the case that $\varepsilon = 0$ and $\gamma = 0$ which means no branching.

Example 1.1. Consider the diffusive part of the particles defined as follows:

$$dx_u^{k,i} = \int_{\mathbb{R}} p_k(y, x_u^{k,i}) W(dy, du)\,, \tag{1.21}$$

where $x_u^{k,i}$ represents ith particle's spatial location, W is a Brownian sheet and $p_k(x,y)$ is the heat kernel defined by (1.19). According to Wang (Lemma 1.3 of Ref. 9), for any integer m and any initial conditions $\{x_0^{k,i} : i = 1, \ldots, m\}$ with $x_0^{k,i} \neq x_0^{k,j}$ if $i \neq j$, the stochastic equations

$$dx_t^{k,i} = \int_{\mathbb{R}} p_k(y, x_t^{k,i}) W(dy, dt)\,, \quad t \geq 0, i = 1, 2, \ldots, m \tag{1.22}$$

have unique strong solutions $\{x_t^{k,i} : t \geq 0\}$ and $\{(x_t^{k,1}, \ldots, x_t^{k,m}) : t \geq 0\}$ is an m-dimensional diffusion process which is governed by the differential operator

$$\bar{G}_0^{m,k} := \frac{1}{2} \sum_{i=1}^{m} \bar{\rho}_k(0) \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i,j=1, i\neq j}^{m} \bar{\rho}_k(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j}\,, \tag{1.23}$$

where

$$\bar{\rho}_k(x) := \int_{\mathbb{R}} p_k(x,y) p_k(0,y) dy\,. \tag{1.24}$$

The question is that

$$\begin{aligned} \bar{\rho}_k(0) &= \int_{\mathbb{R}} p_k(0,y) p_k(0,y) dy \\ &= \int_{\mathbb{R}} \frac{1}{2\pi(1/k)} \exp\left\{-\frac{y^2}{(1/k)}\right\} dy \\ &= [2\sqrt{\pi(1/k)}]^{-1} \int_{\mathbb{R}} \frac{1}{\sqrt{\pi(1/k)}} \exp\left\{-\frac{y^2}{(1/k)}\right\} dy \\ &= [2\sqrt{\pi(1/k)}]^{-1} \to \infty \quad \text{as } k \to \infty\,. \end{aligned} \tag{1.25}$$

Thus, as $k \to \infty$ the generator for the limiting finite particle system has no definition. In the usual sense there is no way to discuss the limiting branching particle system and the corresponding limiting superprocess. Therefore, $\sup_{k\geq 1} \bar{\rho}_k(0) < \infty$ is a necessary condition. We have following main result.

Theorem 1.1. *Let μ be a finite measure on $\mathbb{R}$, $\varepsilon \in \mathbb{R} - \{0\}$ a fixed constant, and $\{g_k \in C_\partial^2(\mathbb{R}) : k \geq 1\}$ be a convergent sequence which converges to a singular function with singular point* 0. *Let $\mathcal{L}_k^\varepsilon$ denote the generator defined by* (1.11) *with*

$$\rho_k(x) := \int_{\mathbb{R}} g_k(x-y) g_k(y) dy\,. \tag{1.26}$$

Then, for any $k \geq 1$, the $(\mathcal{L}_k^\varepsilon, \delta_\mu)$-MP has a unique solution which is a measure-valued diffusion process. Let $\mu_t^{\varepsilon,k}$ denote the unique solution of the $(\mathcal{L}_k^\varepsilon, \delta_\mu)$-MP with sample paths in $C([0,\infty), E)$. Then, as $k \to \infty$ $\{\mu_t^{\varepsilon,k} : k \geq 1\}$ converge to a diffusion process μ_t^ε which is a super-Brownian motion. Furthermore, in the limiting singular case, the diffusive part of the particle system has the following representation:

$$dx_i^{n,\infty}(t) = \sqrt{\rho(0)}d\tilde{B}_t^i + \varepsilon dB_t^i\,, \tag{1.27}$$

where $x_i^{n,\infty}(t)$ is the location of ith limiting particle at time t as $k \to \infty$, $\{\tilde{B}_t^i : t \geq 0, i \geq 1\}$ and $\{B_t^i : t \geq 0, i \geq 1\}$ are independent one-dimensional Brownian motions.

2. Proof of the Main Result

Lemma 2.1. *Let ρ_k be the function defined by* (1.26) *and the sequence of functions of $\{g_k\}$ be a convergent sequence with point $y = 0$ (see Definition* 1.2*). Then, for any $x \neq 0$, $\lim_{k\to\infty} \rho_k(x) = 0$.*

Proof. First for any $\delta > 0$ let $O(x,\delta) = \{y \in \mathbb{R} : |x-y| < \delta\}$ be the ball at x. Then, for any $|x| \neq 0$ and taking $\delta < |x|/2$ we have

$$\begin{aligned}
\rho_k(x) &= \int_{\mathbb{R}} g_k(x-y)g_k(y)dy \\
&= \int_{\mathbb{R}-O(x,\delta)-O(0,\delta)} g_k(x-y)g_k(y)dy + \int_{O(x,\delta)} g_k(x-y)g_k(y)dy \\
&\quad + \int_{O(0,\delta)} g_k(x-y)g_k(y)dy \\
&\leq \sqrt{\int_{\mathbb{R}-O(x,\delta)} g_k^2(x-y)dy}\sqrt{\int_{\mathbb{R}-O(0,\delta)} g_k^2(y)dy} \\
&\quad + \sup_{y\in O(x,\delta)} |g_k(y)| \int_{O(x,\delta)} |g_k(y-x)|dy \\
&\quad + \sup_{y\in O(0,\delta)} |g_k(y-x)| \int_{O(0,\delta)} |g_k(y)|dy \\
&\leq \left\{\sqrt{\sup_{y\in\mathbb{R}-O(x,\delta)} |g_k(x-y)| \int_{\mathbb{R}-O(x,\delta)} |g_k(x-y)|dy}\right\} \\
&\quad \times \left\{\sqrt{\sup_{y\in\mathbb{R}-O(0,\delta)} |g_k(y)| \int_{\mathbb{R}-O(0,\delta)} |g_k(y)|dy}\right\} \\
&\quad + \sup_{y\in O(x,\delta)} |g_k(y)|G + \sup_{y\in O(0,\delta)} |g_k(y-x)|G \to 0\,,
\end{aligned} \tag{2.28}$$

since

$$\lim_{k\to\infty}\sup_{y\in O(x,\delta)}|g_k(y)|=0\,,\quad \lim_{k\to\infty}\sup_{y\in O(0,\delta)}|g_k(y-x)|=0\,,$$

and

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}-O(0,\delta)}|g_k(y)|=0\,,\quad \lim_{k\to\infty}\sup_{y\in\mathbb{R}-O(x,\delta)}|g_k(y-x)|=0\,,$$

and (1.2). □

Lemma 2.2. *Let $\varepsilon\in\mathbb{R}-\{0\}$ be a fixed constant and ρ_k be defined by* (1.26). *Define*

$$G_\varepsilon^{m,k}:=\frac{1}{2}\sum_{i=1}^m\varepsilon^2\frac{\partial^2}{\partial x_i^2}+\frac{1}{2}\sum_{i,j=1}^m\rho_k(x_i-x_j)\frac{\partial^2}{\partial x_i\partial x_j}\,. \tag{2.29}$$

Then, there exists a unique $G_\varepsilon^{m,k}$-diffusion measure $\{\mathbb{P}_x, x\in\mathbb{R}^m\}$ on the m-dimensional Wiener space in the definition of Ikeda–Watanabe.[4] *The corresponding Feller semigroup $S_t^{m,k}$ is defined by:*

$$S_t^{m,k}f(\xi):=\int_{\mathbb{R}^m}Z^{m,k}(\eta,\xi,t)f(\eta)d\eta\quad \xi=(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m\,, \tag{2.30}$$

where $Z^{m,k}(\eta,\xi,t)$ is the transition density function. Let D be the set of points in $\mathbb{R}^m$ with all coordinates distinct and

$$Z^m(\eta,\xi,t):=\frac{1}{[2\pi t(\rho(0)+\varepsilon^2)]^{m/2}}\times\exp\left\{-\frac{1}{2t}\sum_{i=1}^m\frac{(\eta_i-\xi_i)^2}{\rho(0)+\varepsilon^2}\right\}. \tag{2.31}$$

Then, for each $\xi\in D$ we have

$$Z^{m,k}(\eta,\xi,t)\to Z^m(\eta,\xi,t) \tag{2.32}$$

pointwise on $(0,\infty)\times\mathbb{R}^m$ as $k\to\infty$. Furthermore, we have the following inequality

$$|Z^{m,k}(\eta,\xi,t)|\le c_0t^{-m/2}\exp\left(-c_1\frac{|\eta-\xi|^2}{t}\right), \tag{2.33}$$

where c_0 and c_1 are two positive constants which are independent of k.

Proof. The proof of this lemma is just application of the results of (p. 357–363 of Ref. 6). Define

$$\begin{aligned}Z_0^{m,k}(\eta,\xi,t):=&\frac{1}{[2\pi t]^{m/2}(\det A_{m,k}(\xi))^{1/2}}\\&\times\exp\left\{-\frac{1}{2t}\sum_{i,j=1}^m A_{m,k}^{(i,j)}(\xi)(\eta_i-\xi_i)(\eta_j-\xi_j)\right\},\end{aligned} \tag{2.34}$$

where $A_{m,k}(\xi)$ is the covariance matrix defined by

$$A_{m,k}(\xi) = (a_k^{i,j}(\xi)) := \begin{pmatrix} \rho_k(0)+\varepsilon^2 & \cdots & \rho_k(\xi_m-\xi_1) \\ \vdots & \vdots & \vdots \\ \rho_k(\xi_1-\xi_m) & \cdots & \rho_k(0)+\varepsilon^2 \end{pmatrix}, \tag{2.35}$$

$$A_{m,k}^{-1}(\xi) = (A_{m,k}^{(i,j)}(\xi)),$$

is the inverse matrix of $A_{m,k}(\xi)$. According to (11.3) of Ref. 6, we have

$$|D_t^r D_x^s Z_0^{m,k}(x,\xi,t)| \le a_0 t^{-m/2-r-s/2} \exp\left(-a_1 \frac{|x-\xi|^2}{t}\right), \tag{2.36}$$

where D_x^s is the s order partial differential operator. Since $A_{m,k}(\xi)$ is uniformly positive definite in (k,ξ) and $\rho_k(x)$ is uniformly bounded in (k,x), the constants a_0 and a_1 are independent of k. Define

$$K^k(\eta,\xi,t) := \sum_{i,j=1}^m [a_k^{i,j}(\xi) - a_k^{i,j}(\eta)] \frac{\partial^2 Z_0^{m,k}(\eta,\xi,t)}{\partial\eta_i \partial\eta_j}, \tag{2.37}$$

$$K_n^k(\eta,\xi,t) := \int_0^t ds \int_{\mathbb{R}^m} K^k(\eta,y,t-s) K_{n-1}^k(y,\xi,s)dy, \tag{2.38}$$

$$Q^k(\eta,\xi,t) := \sum_{n=1}^\infty (-1)^n K_n^k(\eta,\xi,t). \tag{2.39}$$

By (11.25) of Ref. 6, the above series converges uniformly for $t > 0$. According to (11.13) of Ref. 6, the transition density function is constructed as follows:

$$Z^{m,k}(\eta,\xi,t) = Z_0^{m,k}(\eta,\xi,t) + \int_0^t ds \int_{\mathbb{R}^m} Z_0^{m,k}(\eta,y,t-s) Q^k(y,\xi,s)dy. \tag{2.40}$$

Then, since (11.25), (11.26) of Ref. 6 hold with constants in the inequalities being independent of k, for any η, $\xi \in \mathbb{R}^m$ and for any given $\delta > 0$, there exists a $u > 0$ such that

$$\int_0^u ds \int_{\mathbb{R}^m} |Z_0^{m,k}(\eta,y,t-s) Q^k(y,\xi,s)|dy \le \delta$$

and

$$\int_{t-u}^t ds \int_{\mathbb{R}^m} |Z_0^{m,k}(\eta,y,t-s) Q^k(y,\xi,s)|dy \le \delta$$

hold for all k by (11.3) and (11.26) of Ref. 6. Thus, (2.32) follows from Lebesgue dominated convergence theorem and (2.33) follows from (13.1) of Ref. 6. □

Lemma 2.3. *Let μ be a finite measure on $\mathbb{R}$ and $\{\mu_t^{\varepsilon,k} : t \ge 0\}$ be a measure-valued diffusion process which has sample paths in $C([0,\infty), E)$ and is the unique solution to the $(\mathcal{L}_k^\varepsilon, \delta_\mu)$-MP. Then, $\{\langle 1, \mu_t^{\varepsilon,k}\rangle : t \ge 0\}$ is a tight sequence.*

Proof. Here we need the extensions to $\bar{\mathbb{R}}$. For more details, the reader is referred to Wang[10] and Lemma 3.4. $\square$

Proof of Theorem 1.1. For any finite measure μ on $\mathbb{R}$, any $k \geq 1$ and any fixed $\varepsilon \in \mathbb{R} - \{0\}$, the existence and uniqueness of the $(\mathcal{L}_k^\varepsilon, \delta_\mu)$-martingale problem and its solution $\mu_t^{\varepsilon,k}$ being a diffusion process are proved in Wang[10] For more general conditions, see Dawson–Li–Wang.[3] By Lemma 2.3, there exists a weak convergence subsequence of $\mu_t^{\varepsilon,k}$. Without loss of generality, here we assume that $\mu_t^{\varepsilon,k} \Rightarrow \mu_t^\varepsilon$. By Skorohod's representation theorem, there exists a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $\mu_t^{\varepsilon,k} \to \mu_t^\varepsilon$ almost surely on this new probability space. By Lemma 3.4 of Ref. 10, we get the uniform integrability of $\langle \phi, \mu_t^{\varepsilon,k} \rangle$ for any given test function $\phi \in \mathcal{S}(\mathbb{R})$. Thus,

$$\lim_{k\to\infty} \tilde{\mathbb{E}}\langle \phi, \mu_t^{\varepsilon,k} \rangle = \tilde{\mathbb{E}}\langle \phi, \mu_t^\varepsilon \rangle \,. \tag{2.41}$$

For each integer $m \geq 1$, we define an operator for each $f \in C(\mathbb{R}^m)$

$$T_t^m f(\xi) := \int_{\mathbb{R}^m} Z^m(\eta, \xi, t) f(\eta) d\eta \,, \quad \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \,. \tag{2.42}$$

It is easy to check that this is just the Feller semigroup of an m-dimensional generalized Brownian motion with a constant covariance $\rho(0) + \varepsilon^2$. From Lemma 2.2 and Lebesgue dominated convergence theorem and any $f \in C(\mathbb{R}^m)$ we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\langle f, \mu_t^\varepsilon \rangle \\
&\quad = \lim_{k\to\infty} \tilde{\mathbb{E}}\langle f, \mu_t^{\varepsilon,k} \rangle \\
&\quad = \lim_{k\to\infty} \hat{\mathbb{E}}\left[\langle Y^k(t), \mu^{M(t)} \rangle \exp\left\{ \beta \int_0^t M(u)(M(u)-1)du \right\} \right] \\
&\quad = \hat{\mathbb{E}}\left[\langle Y(t), \mu^{M(t)} \rangle \exp\left\{ \beta \int_0^t M(u)(M(u)-1)du \right\} \right] \,,
\end{aligned} \tag{2.43}$$

where $Y_0 = f$,

$$\begin{gathered}
Y^k(t) := S_{t-\tau_n}^{M_{\tau_n},k} \Gamma_n S_{\tau_n - \tau_{n-1}}^{M_{\tau_{n-1}},k} \Gamma_{n-1} \cdots S_{\tau_2-\tau_1}^{M_{\tau_1},k} \Gamma_1 S_{\tau_1}^{M_{\tau_0},k} Y_0 \,, \\
\tau_n \leq t < \tau_{n+1} \,, \quad 0 \leq n \leq M_0 - 1 \,,
\end{gathered} \tag{2.44}$$

and

$$\begin{gathered}
Y(t) := T_{t-\tau_n}^{M_{\tau_n}} \Gamma_n T_{\tau_n - \tau_{n-1}}^{M_{\tau_{n-1}}} \Gamma_{n-1} \cdots T_{\tau_2-\tau_1}^{M_{\tau_1}} \Gamma_1 T_{\tau_1}^{M_{\tau_0}} Y_0 \,, \\
\tau_n \leq t < \tau_{n+1} \,, \quad 0 \leq n \leq M_0 - 1 \,,
\end{gathered} \tag{2.45}$$

where $\{M_t : t \geq 0\}$ is a non-negative integer-valued cádlág Markov process with transition intensities $\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = \beta i(i-1)$ and $q_{i,j} = 0$ for all other pairs (i,j). That is, $\{M_t : t \geq 0\}$ is the well-known Kingman's coalescent process. Define $\tau_0 = 0$, $\tau_{M_0} = \infty$ and $\{\tau_k : 1 \leq k \leq M_0 - 1\}$ to be the sequence of jump times of $\{M_t : t \geq 0\}$. $\{\Gamma_k : 1 \leq k \leq M_0 - 1\}$ is a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\mathbb{P}\{\Gamma_k = \Phi_{i,j} | M(\tau_k -) = l\} = \frac{1}{l(l-1)}\,, \quad 1 \leq i \neq j \leq l\,, \tag{2.46}$$

where $\Phi_{i,j}$, which maps an m-dimensional function f to an $(m-1)$-dimensional function $\Phi_{j,i}f$, is defined by

$$\begin{aligned}&[\Phi_{j,i}f](x_1, \dots, x_{m-2}, y)\\ &\quad := \begin{cases} f(x_1, \dots, x_{j-1}, y, x_j, \dots, x_{i-1}, y, x_i, \dots, x_{m-2}) & \text{if } j < i \\ f(x_1, \dots, x_{i-1}, y, x_i, \dots, x_{j-1}, y, x_j, \dots, x_{m-2}) & \text{if } i < j\,. \end{cases}\end{aligned} \tag{2.47}$$

For each monomial function

$$F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \dots, x_m) \mu^{\otimes m}(d\tilde{x})\,,$$

where $\mu^{\otimes m}(d\tilde{x}) := \mu(dx_1) \times \cdots \times \mu(dx_m)$ (see (2.16) in Ref. 10). For any $\varepsilon \in \mathbb{R} - \{0\}$, define

$$\mathcal{A}^\varepsilon F_f(\nu) := \int_{\mathbb{R}^m} G_\varepsilon^m f(x_1, \dots, x_m) \nu^{\otimes m}(d\tilde{x})\,, \tag{2.48}$$

where

$$G_\varepsilon^m := \frac{1}{2} \sum_{i=1}^m (\rho(0) + \varepsilon^2) \frac{\partial^2}{\partial x_i^2} \tag{2.49}$$

and

$$\mathcal{L}^\varepsilon F_f(\mu) = \mathcal{A}^\varepsilon F_f(\nu) + \mathcal{B} F_f(\nu)\,, \tag{2.50}$$

where

$$\mathcal{B} F_f(\nu) = \beta \sum_{i,j=1, i \neq j}^m [F_\nu(\Phi_{j,i} f) - F_\nu(f)] + \beta m(m-1) F_\nu(f)\,. \tag{2.51}$$

It is obvious that the right-hand side of (2.43) comes from the Feynman–Kac formula and the generator $\mathcal{L}^\varepsilon$ which is just the generator of the super-Brownian motion. (1.27) follows from the martingale representation theorem (see Ref. 4).

3. Comments and Example

Example 3.1. Define

$$q_k(x, y) = \frac{1}{[2\pi(1/k)]^{1/4}} \exp\left\{ -\frac{(x-y)^2}{4(1/k)} \right\}\,. \tag{3.52}$$

Consider the diffusive part of the particles defined as follows:

$$dx_u^{k,i} = \int_{\mathbb{R}} q_k(y, x_u^{k,i}) W(dy, du) + \varepsilon dB_t^i\,, \quad i = 1, \dots, m\,, \tag{3.53}$$

where W is a Brownian sheet, $\{B_t^i; i = 1, \ldots, m\}$ are independent one-dimensional Brownian motions which are independent of W, $\varepsilon \in \mathbb{R} - \{0\}$, and $x_u^{k,i}$ represents ith particle's spatial location at time u, where the sup-index k of $x_u^{k,i}$ is used to indicate that the particles are related to $q_k(\cdot, \cdot)$. It is obvious that for any integer m and any initial conditions $\{x_0^{k,i}: i = 1, \ldots, m\}$ with $x_0^{k,i} \neq x_0^{k,j}$ if $i \neq j$, $\{(x_t^{k,1}, \ldots, x_t^{k,m}): t \geq 0\}$ is an m-dimensional diffusion process which is generated by the differential operator

$$\tilde{G}_\varepsilon^{m,k} := \frac{1}{2} \sum_{i=1}^m \varepsilon^2 \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i,j=1}^m \tilde{\rho}_k(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} \,, \tag{3.54}$$

where

$$\tilde{\rho}_k(x) := \int_{\mathbb{R}} q_k(x, y) q_k(0, y) dy \,. \tag{3.55}$$

In this case,

$$\begin{aligned} \tilde{\rho}_k(0) &= \int_{\mathbb{R}} q_k(0, y) q_k(0, y) dy \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(1/k)}} \exp\left\{ -\frac{y^2}{2(1/k)} \right\} dy \\ &= 1 \quad \text{for any } k \,. \end{aligned} \tag{3.56}$$

Using $p_k(\cdot, \cdot)$ defined by (1.19), we have that the increasing process for each particle's location $x_t^{k,i}$ is

$$\begin{aligned} \langle x_t^{k,i} \rangle &= \int_0^t \int_{\mathbb{R}} p_k(y, x_u^{k,i}) dy du \\ &= t \end{aligned} \tag{3.57}$$

and for any $i \neq j$ the quadratic variational process for the finite particle system $\{x_t^{k,i}: i = 1, \ldots, m\}$ is

$$\begin{aligned} \langle x^{k,i}, x^{k,j} \rangle_t &= \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(1/k)}} \exp\left\{ -\frac{(y - x_u^{k,i})^2 + (y - x_u^{k,j})^2}{4/k} \right\} dy du \\ &= \int_0^t \frac{1}{\sqrt{2\pi(1/k)}} \exp\left\{ -\frac{(x_u^{k,i} - x_u^{k,j})^2}{8/k} \right\} du > 0 \,. \end{aligned} \tag{3.58}$$

As $k \to \infty$, since $x_u^{k,j}$ depends on k, we are not sure whether the above quadratic variational process converges to zero. However, according to the above theorem, we have the following general result. Let $x_t^{\infty,i}$ be the location of ith limiting particle as $k \to \infty$. Then,

$$\langle x_t^{\infty,i}, x_t^{\infty,j} \rangle = 0 \quad \text{if } i \neq j$$

and

$$dx_t^{\infty,i} = \sqrt{\rho(0)} d\tilde{B}_t^i + \varepsilon dB_t^i \,, \quad i = 1, \ldots, m$$

where $\{\tilde{B}_t^i : t \geq 0, i = 1, \ldots, m\}$ are independent one-dimensional Brownian motions which are independent of $\{B_t^i : t \geq 0, i = 1, \ldots, m\}$.

Remark 3.1. For $\varepsilon \neq 0$, by dual representation of the superprocesses we have found a class of singular spacetime Itô's integrals. For $\varepsilon = 0$, due to the degeneracy and coalescence property, there are many unsolved challenging problems.

Remark 3.2. Here we give some comments for the case that $\varepsilon = 0$. Before doing so, we need the following definition.

Coalescence Property. A particle system is said to have *coalescence property* if the particle location processes are diffusion processes and for any two particles either they never separate or they never meet according as they start off from same initial location or not.

For a given initial measure $\mu_0 = \sum_{i=1}^m \frac{1}{\theta^M} \delta_{x_0^{k,i}}$, if there are only n, where $n < m$, different locations, we have

$$\mu_0 = \sum_{i=1}^{m} \frac{1}{\theta^M} \delta_{x_0^{k,i}} = \sum_{i=1}^{n} \frac{k_i}{\theta^M} \delta_{x_i} .$$

where k_i is an integer representing number of particles located at same location x_i, $x_i \neq x_j$ if $i \neq j$. By using (3.55), we define

$$\begin{aligned} \tilde{\mathcal{A}}_k^{\varepsilon} F(\mu) &:= \frac{1}{2} \int_{\mathbb{R}} (\tilde{\rho}_k(0) + \varepsilon^2) \left(\frac{d^2}{dx^2} \right) \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{\rho}_k(x-y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy) . \end{aligned} \tag{3.59}$$

First, for any monomial function

$$F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_N) \mu^{\otimes N}(d\tilde{x})$$

and above $\mu = \sum_{i=1}^m \frac{1}{\theta^M} \delta_{x_0^{k,i}} = \sum_{i=1}^n \frac{k_i}{\theta^M} \delta_{x_i}$ with $x_i \neq x_j$ if $i \neq j$ and $n < m$. Consider diffusion process $\{(x_t^{k,1}, \ldots, x_t^{k,m}) : t \geq 0\}$ which is generated by the differential operator $\tilde{G}_0^{n,k}$ defined by (3.54) with $\varepsilon = 0$ and initial state $\{(x_0^{k,1}, \ldots, x_0^{k,m})$. For any $1 \leq i < j \leq m$, define $\eta_t := x_t^{k,i} - x_t^{k,j}$. Then, $\{\eta_t\}$ is a diffusion process with state space $\mathbb{R}$, absorbing state 0 and generator

$$\mathcal{G}_0 f(y) = (\tilde{\rho}_k(0) - \tilde{\rho}_k(y)) f''(y) , \quad f \in C_b^{\infty}(\mathbb{R}) ,$$

where $\tilde{\rho}_k(x)$ is defined by (3.55).

From Feller's criterion of accessibility, the probability that η reaches 0 is 0 or 1 according as

$$\int_0^1 \frac{y}{(\tilde{\rho}_k(0) - \tilde{\rho}_k(y))} dy$$

is ∞ or $< \infty$. It is easy to check that $\tilde{\rho}_k(\cdot)$ is non-negative definite, then by the Bochner–Khinchin theorem there is a probability distribution function $F(\cdot)$ such that

$$\begin{aligned} 0 \le 1 - \frac{\tilde{\rho}_k(y)}{\tilde{\rho}_k(0)} &= \int_{\mathbb{R}} \{1 - \cos(xy)\} dF(x) \\ &\le \int_{\mathbb{R}} \frac{1}{2}(xy)^2 dF(x) = \frac{1}{2\tilde{\rho}_k(0)} y^2 |\tilde{\rho}_k''(0)|\,. \end{aligned}$$

Hence we get

$$0 \le \sup_y \frac{(\tilde{\rho}_k(0) - \tilde{\rho}_k(y))}{y^2} \le \frac{1}{2}|\tilde{\rho}_k''(0)|\,. \tag{3.60}$$

Since q_k is smooth and $\tilde{\rho}_k''(0)$ is finite, state 0 is inaccessible. Thus, coalescence property holds and $\{(x_t^{k,1}, \ldots, x_t^{k,m}) : t \ge 0\}$ is an n-dimensional diffusion process $(n < m)$. Now we change the form of $\tilde{\mathcal{A}}_k^0 F_f(\mu)$ as follows.

$$\begin{aligned} \tilde{\mathcal{A}}_k^0 F_f(\mu) := &\frac{1}{2}\tilde{\rho}_k(0) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i=1}^{N} f_{ii}''(y_1, \ldots, y_N) \mu^{\otimes N}(d\tilde{y}) \\ &+ \frac{1}{2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1; i\ne j}^{N} \tilde{\rho}_k(y_i - y_j) f_{ij}''(y_1, \ldots, y_N) \mu^{\otimes N}(d\tilde{y}) \\ = &\frac{1}{2} \sum_{i=1}^{n} \tilde{\rho}_k(0) \frac{\partial^2}{\partial x_i^2} g(x_1, \ldots, x_n) \quad (\varepsilon = 0 \text{ and coalescence property}) \\ &+ \frac{1}{2} \sum_{i,j=1; i\ne j}^{n} \tilde{\rho}_k(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} g(x_1, \ldots, x_n) \\ &- \frac{1}{2\theta^M} \tilde{\rho}_k(0) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1, i\ne j}^{N} f_{ij}''(y_1, \ldots, y_N) \mu^{\otimes N-1}(d\tilde{y}) \\ = &\tilde{G}_0^{n,k} g(x_1, \ldots, x_n) \\ &- \frac{1}{2\theta^M} \tilde{\rho}_k(0) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1, i\ne j}^{N} f_{ij}''(y_1, \ldots, y_N) \mu^{\otimes N-1}(d\tilde{y})\,, \end{aligned} \tag{3.61}$$

where

$$g(x_1, \ldots, x_n) = \left(\frac{1}{\theta^M}\right)^N \sum_{l_1, \ldots, l_N = 1}^{n} k_{l_1} \cdots k_{l_N} f(x_{l_1}, \ldots, x_{l_N})\,,$$

and $\tilde{G}_0^{n,k}$ is defined by (3.54) with $\varepsilon = 0$.

However, for the same monomial function

$$F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_N) \mu^{\otimes N}(d\tilde{x}),$$

and the same $\mu = \sum_{i=1}^m \frac{1}{\theta^M} \delta_{x_0^{k,i}} = \sum_{i=1}^n \frac{k_i}{\theta^M} \delta_{x_i}$ with $x_i \neq x_j$ if $i \neq j$ and $n < m$, for $\varepsilon \in \mathbb{R} - \{0\}$, which is the same as before consider diffusion process $\{(x_t^{k,1}, \ldots, x_t^{k,m}) : t \geq 0\}$ which is generated by the differential operator $\tilde{G}_\varepsilon^{n,k}$ defined by (3.54) with $\varepsilon \neq 0$ and initial state $\{(x_0^{k,1}, \ldots, x_0^{k,m})$. For any $1 \leq i < j \leq m$, define $\xi_t := x_t^{k,i} - x_t^{k,j}$. Then, $\{\xi_t\}$ is a diffusion process with state space $\mathbb{R}$ and generator

$$\mathcal{G}_\varepsilon f(y) = (\tilde{\rho}_k(0) + \varepsilon^2 - \tilde{\rho}_k(y)) f''(y), \quad f \in C_b^\infty(\mathbb{R}),$$

where $\tilde{\rho}_k(x)$ is defined by (3.55). From this generator, we see that 0 is not a boundary point. Therefore, we have

$$\begin{aligned}
\tilde{\mathcal{A}}_k^\varepsilon F_f(\mu) := {} & \frac{1}{2}(\tilde{\rho}_k(0) + \varepsilon^2) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i=1}^N f''_{ii}(y_1, \ldots, y_N) \mu^{\otimes N}(d\tilde{y}) \\
& + \frac{1}{2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1; i\neq j}^N \tilde{\rho}_k(y_i - y_j) f''_{ij}(y_1, \ldots, y_N) \mu^{\otimes N}(d\tilde{y}) \\
& \times (\varepsilon \neq 0,\ 0 \text{ is not an absorbing boundary}) \\
= {} & \frac{1}{2} \sum_{i=1}^m (\tilde{\rho}_k(0) + \varepsilon^2) \frac{\partial^2}{\partial x_i^2} g(x_1, \ldots, x_m) \\
& + \frac{1}{2} \sum_{i,j=1; i\neq j}^m \tilde{\rho}_k(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} g(x_1, \ldots, x_m) \\
& - \frac{1}{2\theta^M}(\tilde{\rho}_k(0) + \varepsilon^2) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1, i\neq j}^N f''_{ij}(y_1, \ldots, y_N) \mu^{\otimes N-1}(d\tilde{y}) \\
= {} & \tilde{G}_\varepsilon^{m,k} g(x_1, \ldots, x_m) \\
& - \frac{1}{2\theta^M}(\tilde{\rho}_k(0) + \varepsilon^2) \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1, i\neq j}^N f''_{ij}(y_1, \ldots, y_N) \mu^{\otimes N-1}(d\tilde{y}).
\end{aligned} \tag{3.62}$$

where

$$g(x_1, \ldots, x_m) = \left(\frac{1}{\theta^M}\right)^N \sum_{l_1, \ldots, l_N = 1}^m f(x_{l_1}, \ldots, x_{l_N}).$$

Compare (3.61) with (3.62), since $n < m$, we see that by the notation of Theorem 1.1, $\{\mu_t^{0,k}\}$ cannot converge to a super-Brownian motion due to the coalescence property.

Acknowledgment

I would like to take this opportunity to express my sincere gratitude to Prof. Donald A. Dawson and Prof. Zenghu Li for their helpful suggestions and valuable discussions.

References

1. P. Antosik, J. Mikusiński and R. Sikorski, *Theory of Distributions, the Sequential Approach* (PWN, 1973).
2. D. A. Dawson, Measure-Valued Markov Processes, *Lecture Notes in Math.*, Vol. 1541 (Springer, 1993), pp. 1–260.
3. D. A. Dawson, Z. Li and H. Wang, *Superprocesses with dependent spatial motion and general branching densities*, *Electronic J. Probab.* **6** (2001) 1–33.
4. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North-Holland, 1981).
5. N. Konno and T. Shiga, *Stochastic partial differential equations for some measure-valued diffusions*, *Probab. Th. Rel. Fields* **79** (1988) 201–225.
6. O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type* (American Mathematical Society, 1968).
7. J. B. Walsh, *An Introduction to Stoshastic Partial Differential Equations*, *Lecture Notes in Math.*, Vol. 1180 (1986), pp. 265–439.
8. H. Wang, *Interacting branching particle systems and superprocesses*, Ph.D thesis, Carleton University, Canada, 1995.
9. H. Wang, *State classification for a class of measure-valued branching diffusions in a Brownian medium*, *Probab. Th. Rel. Fields* **109** (1997) 39–55.
10. H. Wang, *A class of measure-valued branching diffusions in a random medium*, *Stochastic Anal. Appl.* **16** (1998) 753–786.