IMA Journal of Management Mathematics (2006) **17**, 289−303
doi:10.1093/imaman/dpi044
Advance Access publication on February 1, 2006

A calibrated scenario generation model for heavy-tailed risk factors

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[Received on 3 October 2003; revised on 5 October 2005; accepted on 18 November 2005]

In this paper, a calibrated scenario generation model for multivariate risk factors with heavy-tailed distributions is developed. This model includes the standard and classical model of scenario generation developed by J. P. Morgan as a special case. A rotation method is introduced to preserve the correlation information between risk factors, and a mixture of normal distributions is used to model and fit each marginal heavy-tailed distribution. Based on the scenario generation, a non-parametric method is applied to estimate the extreme value-at-risk and value-at-risk confidence interval of a portfolio with heavy-tailed distribution.

Keywords: VaR; VaR CI; heavy tails; non-parametric method; Monte Carlo simulation; scenario generation.

1. Introduction

Portfolio value-at-risk (VaR) and the value-at-risk confidence interval (VaR CI) are fundamental tools in risk management. By definition, a portfolio is a linear combination of shares of financial instruments. A mutual fund is a typical example of a portfolio. To estimate portfolio VaR and VaR CI, J. P. Morgan initially developed a widely accepted standard and classical model of scenario generation and portfolio VaR and VaR CI estimation (J. P. Morgan/Reuters, 1996; Morgan, 1997). This model was later updated by RiskMetrics (Mina & Xiao, 2001). In their model a portfolio is treated as a function of risk factors, where the risk factors include interest rates, foreign exchange rates, equities, commodities, etc., and the log-returns of the risk factors are assumed normally distributed. However, the empirical distribution of short-term log-returns of risk factors usually demonstrate heavy-tailed behaviour. This is true and widely acknowledged for the daily log-returns. Therefore, their model may not work for heavy-tailed cases. This raises a challenging problem of how to generalize their model

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to handle heavy-tailed cases. The main purpose of this paper is to present some ideas to attack this problem.

In the nineties of the last century, we have seen the deficiencies of normal and light-tailed distribution models in risk management. Heavy-tailed phenomena have brought wide attention. (For example, see Heston, 1993; Ritchken & Trevor, 1999; Heston & Nandi, 2000; de Haan & Peng, 1998; de Haan & Pereira, 1999; Embrechts *et al.*, 1997; Feigin & Resnick, 1997; Resnick, 1987; Hull & White, 1998; Huisman *et al.*, 1997; Su & Fleisher, 1998; Tucker *et al.*,1994; Koedijk, 1990; Ho *et al.*, 2000; Muller *et al.*, 1998; Sornette *et al.*, 2000. Here we name only a few.) Daily log-return financial data, credit risk involved data and energy price data often appear to have heavy tails. The empirical study of historical data suggests the rejection of the hypothesis of normal or even log-normal distributions (see Upton & Shannon, 1979; Tucker *et al.*, 1994). Thus, many existing models based on normal or log-normal assumptions are inadequate. For example, the famous Black-Scholes formula is simply not valid for heavy-tailed cases.

To develop risk management tools and handle non-normality in observed time series, many new ideas have been proposed. There are a number of papers applying extreme value theory (EVT) to estimate portfolio VaR and extreme VaR. The basic idea of the EVT techniques is to first estimate the tail index of a heavy-tailed distribution of the portfolio data using, e.g. a tail index estimator such as the Hill estimator, the Pickand estimator and the Deckers-Einmahl-de Haan estimator, and then to estimate the VaR and extreme VaR, since they are functions of the tail index. However, EVT techniques are not suitable for all heavy-tailed distributions. To explain this point, we need to introduce the following new terms. We call a distribution weakly heavy tailed if the kurtosis (see (2.10) for the definition) of the distribution exists and is bigger than 3 (the kurtosis of any normal distribution is always equal to 3). We call a distribution strongly heavy tailed if the distribution does not have a finite kurtosis. From examples in Embrechts (1997), EVT techniques work reasonably well for strongly heavy-tailed data such as insurance data or other data involving big jumps. However, from simulation results, the daily log-returns of most financial risk factors have symmetric weakly heavy-tailed distributions. The EVT tail index estimators are usually inappropriate for weakly heavy-tailed distributions. For instance, we can generate random numbers having t-distribution with degree equal to 20, where the sample size can be as large as we like. In this case, the value of the Hill estimator should be close to 0.05 (cf. Embrechts, 1997, pp. 330–334). However, we find it is impossible to conclude this from the Hill plot. EVT also has several other drawbacks.

One obstacle in the EVT techniques is that it is difficult to make an optimal choice for the cut-off parameter or sample fraction. Recently, Danielsson *et al.* (2001) proposed some ideas to approach this problem. Another problem is that extreme values come from rare events; thus, the size of extreme observations is often small even though the whole sample size is quite large. It is thus difficult to obtain precise parameter estimation in small-sample cases, therefore raising a problem for the applicability of the EVT techniques. From the consideration of scenario generation, EVT is only concerned with the tail part information of an empirical distribution and ignores the information in the remaining parts of a distribution. Thus, the EVT method only calibrates information on the tail part of an empirical distribution and therefore is clearly not directly applicable to the scenario generation.

The traditional approach has assumed that, although unconditional returns are not normal, properly conditioned returns are normal. In the GARCH or stochastic volatility models, the returns are normal conditioned on knowing the current variance. The returns are normal in jump-diffusion models conditioned on no-jumps. In Markov-switching models, the returns are normal conditioned on knowing the current state. Hull & White (1998) proposed an interesting alternate method to handle non-normality. However, we have several questions concerning their model after analysing their techniques. First, in their model the returns are normal after a functional transformation. Of course, a multivariate normal distribution is much easier to handle. However, for normally or Gaussian-distributed risk factors, their independence is equivalent to zero correlation, where zero correlation means that any two different risk factors have zero correlation coefficient. This is not the case for non-normally distributed risk factors. Therefore, the functional transformation apparently causes the loss of some correlation information if the original distribution of the risk factors is not normally distributed. Second, Hull & White (1998) have used a fractile matching method to calibrate each non-normal marginal distribution. We know that generally four fractiles cannot completely determine a normal distribution. For a non-normal distribution, the situation even becomes more complicated. Third, in Hull & White (1998) the portfolio VaR is estimated by an approximation of the first two moments of the corresponding Taylor expansion. This may result in insufficient information to estimate the extreme VaR of the risk factors with heavy-tailed distribution if the sample size is not large enough, since extreme values are rare events. These problems provide partial motivation for us to generalize Hull–White's model. We will generalize both Hull–White's model and the standard method of scenario generation developed by J. P. Morgan/Reuters (1996), and later updated by RiskMetrics (Mina & Xiao, 2001), to replicate the heavy-tailed risk factors. In our model, we assume that risk factors have weakly heavy-tailed distribution with finite sixth moment. We use a rotation method to preserve the risk factors' correlation information and use a precise moment calibration system of equations to calibrate each non-normal marginal distribution. The first two moments completely determine a normal distribution. Our model calibrates the first six moments. It is our belief that when the variance, skewness, kurtosis and up to the sixth moment of a scenario distribution match those of a heavy-tailed data distribution we can obtain a very high accuracy of replication of the heavy-tailed distribution of a market risk factor.

We will use a non-parametric method to estimate the portfolio extreme VaR and VaR CI. Since we can have arbitrarily large scenario size, we can accurately estimate portfolio extreme VaR as long as the scenarios are generated with a precise calibration of the parameters. Specifically, the paper's main ideas are as follows. First, in order to estimate portfolio extreme VaR and VaR CI with heavy-tailed risk factors by non-parametric method, we must obtain a large sample. Thus, the replication of risk factors and scenario generation is one of the best ways to approach this problem since a portfolio is a function of some risk factors. Second, to preserve correlation information of the risk factors, we use a rotation method to transform the data covariance matrix into a diagonal matrix; this is possible since the data covariance matrix is a real, symmetric matrix. In other words, we can find another orthonormal basis. With respect to this orthonormal basis, the data covariance matrix becomes diagonal. Third, on the new orthonormal basis, similar to the method used in Hull & White (1998), we use a functional transformation to transform each non-normal marginal distribution to a normal marginal distribution. Therefore, after the functional transformation, on the new orthonormal basis each marginal distribution becomes a normal distribution. Moreover, we assume that the data covariance matrix is still a diagonal matrix after a functional transformation. This is especially true if the original multivariate distribution is an elliptic contoured distribution (see Fang & Zhang, 1990, for the definition). Thus, after the functional transformation with respect to the new orthonormal basis, the transformed risk factors are independent and have normal distribution. Based on this analysis, we first generate 1D normally distributed scenarios independently n times. Then, by the inverse functional transformation and the inverse rotation transformation we obtain the scenarios that simulate and replicate the market risk factors.

Now let us introduce our model.

2. Preliminary and model

The VaR[1] of a portfolio for a given confidence level θ is defined as a θ quantile, denoted by Q_θ. Formally, we can define a θ quantile Q_θ as follows.

$$\mathbb{P}(V_0 - V_t \geqslant Q_\theta) = 1 - \theta, \tag{2.1}$$

where V_0 is the current portfolio value, V_t is the portfolio value at future time t and $\mathbb{P}$ is the market probability. For related terms, the reader is referred to Casella & Berger (1990), Pritsker (1997), J. P. Morgan/Reuters (1996), Morgan (1997) and Jorion (2000). Without loss of generality, we can assume that the values of any risk factors are positive. This class of risk factors includes interest rates, foreign exchange rates, equities, commodities and so on. Generally, a portfolio contains multiple risk factors. In this paper, we only consider one-step Monte Carlo scenario generation for n risk factors. The one step is just a unit of time. Since the class of mixtures of normal distributions includes normal distributions as a subclass, the one step can be 1 day or more than 1 day. Therefore, our model includes RiskMetrics' model as a special case. For the convenience of description, in this paper we describe the 1-day scenario generation and our model validation will also be developed based on daily log-return data. Now, we begin by introducing some basic notations and definitions. In our model, the dynamics of the n-dimensional vector of risk factors $\boldsymbol{R}$ is assumed as follows.

$$\log \boldsymbol{R}(t+1) - \log \boldsymbol{R}(t) = \boldsymbol{\xi}, \tag{2.2}$$

where

$$\log \boldsymbol{R}(t+1) = \begin{pmatrix} \log r_1(t+1) \\ \log r_2(t+1) \\ \vdots \\ \log r_n(t+1) \end{pmatrix}, \quad \log \boldsymbol{R}(t) = \begin{pmatrix} \log r_1(t) \\ \log r_2(t) \\ \vdots \\ \log r_n(t) \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$

are n-dimensional vectors. Thus, we have assumed that there are n different risk factors. $\boldsymbol{R}(t)$ is today's value of the risk factors and $\boldsymbol{R}(t+1)$ is tomorrow's value of the risk factors. $\boldsymbol{\xi}$ is an n-dimensional random vector. Here, $\boldsymbol{\xi}$ may have a heavy-tailed distribution. The daily log-return of the ith risk factor by our notation is

$$\xi_i \equiv \log \frac{r_i(t+1)}{r_i(t)}. \tag{2.3}$$

Let $F(x_1, \ldots, x_n, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ denote the distribution of $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^\top$ with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, where the superscript $\top$ denotes the operation of transposing. Since $\boldsymbol{\Sigma}$ is a real and symmetric matrix, there exists an orthonormal matrix $\boldsymbol{U}$ such that

$$\boldsymbol{U}^\top \boldsymbol{\Sigma} \boldsymbol{U} = \boldsymbol{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n). \tag{2.4}$$

Define $\boldsymbol{G} = \boldsymbol{U}^\top \boldsymbol{\xi}$. Then, $\boldsymbol{\Lambda}$ is the covariance matrix of $\boldsymbol{G} = (g_1, \ldots, g_n)^\top$.

Basic Assumption: In our model, we assume that the distribution of $\boldsymbol{\xi}$ does not depend on time t and $\{g_1, \ldots, g_n\}$ are independent.

REMARK 2.1 According to the definition of elliptic contoured distributions defined in Fang & Zhang (1990), if $F(x_1, \ldots, x_n, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is an elliptic contoured distribution, then, $\{g_1, \ldots, g_n\}$ are independent.

[1]In the market or financial literature, the VaR of a portfolio is defined as an amount of money such that with a small probability the portfolio will lose more than that amount over a given period of time. Therefore, the market VaR is just the absolute value of our defined VaR.

Elliptic contoured distributions include multivariate normal distributions, multivariate uniform distributions and multivariate t-distributions as special cases. (See Fang & Zhang, 1990, for other cases.)

Let F_i denote the distribution of g_i. Then, the random variable $F_i(g_i)$ has a uniform distribution on [0, 1]. Define $e_i = N^{-1}(F_i(g_i))$, where N^{-1} is the inverse function of the standard normal distribution function. Then, e_i has a standard normal distribution. Based on our basic assumption, $\{e_1, \dots, e_n\}$ are independent random variables.

Let $x_{i1}, x_{i2}, \dots, x_{im}$ be the observations of the random variable ξ_i or the daily log-returns of the ith risk factor. Here, m is the number of observation days or the sample size. Define

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}. \tag{2.5}$$

Here without loss of generality, we can assume that the log-returns have mean zero. Otherwise, a simple transformation can change it into this situation. The covariance matrix of the log-return of the risk factors can be estimated by the method of moments as follows.

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{m}\boldsymbol{X}\boldsymbol{X}^{\top}. \tag{2.6}$$

Since $\hat{\boldsymbol{\Sigma}}$ is a real, symmetric matrix, we can find an $n \times n$ orthonormal matrix $\hat{\boldsymbol{U}} = (u_{ij})$ such that

$$\hat{\boldsymbol{U}}^{\top}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{U}} = \hat{\boldsymbol{\Gamma}} \equiv \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n \end{pmatrix} \tag{2.7}$$

is a diagonal matrix with $\gamma_i \geqslant 0$. Define

$$\boldsymbol{Y} = (y_{ij}) = \hat{\boldsymbol{U}}^{\top}\boldsymbol{X}, \quad y_{ij} = \sum_{k=1}^{n} u_{ik}x_{kj}. \tag{2.8}$$

Then,

$$\frac{1}{m}\boldsymbol{Y}\boldsymbol{Y}^{\top} = \hat{\boldsymbol{\Gamma}}. \tag{2.9}$$

This means that after rotation, the new data has a diagonal covariance matrix. We will see that this transformation is the key to success in the calibrations of the covariance matrix and other parameters. For our convenience, $y_{i1}, y_{i2}, \dots, y_{im}$ are called the observations or the sample of the ith transformed risk factor. Later on we will need $\hat{\boldsymbol{U}}$ to rotate back the generated scenarios in order to get scenarios with all the desired calibration properties. Let

$$\kappa_i = \frac{\frac{1}{m}\sum_{k=1}^{m}(y_{ik} - \bar{y}_i)^4}{\left\{\frac{1}{m}\sum_{k=1}^{m}(y_{ik} - \bar{y}_i)^2\right\}^2} \tag{2.10}$$

be the data kurtosis of the transformed ith risk factor, where

$$\bar{y}_i = \frac{1}{m}\sum_{k=1}^{m} y_{ik}. \tag{2.11}$$

For a given random variable ϑ, if it has finite fourth moment, then its kurtosis is defined by

$$\kappa(\vartheta) = \frac{\mathbb{E}(\vartheta - \mathbb{E}\vartheta)^4}{\{\mathbb{E}(\vartheta - \mathbb{E}\vartheta)^2\}^2}, \tag{2.12}$$

where $\mathbb{E}\vartheta$ is the expectation of the random variable ϑ. In our experience, the daily log-return data always have a symmetrical empirical distribution. Thus, we do not introduce skewness here. Now, let us introduce a 1D mixture of normal distributions (MND).

DEFINITION 2.1 We say that a random variable ζ has a 1D MND if its density function $f_\zeta(x)$ exists and has the following representation: There exists a number p satisfying $0 \leqslant p \leqslant 1$ such that

$$f_\zeta(x) = pf(x, \mu_\alpha, \sigma_\alpha) + (1-p)f(x, \mu_\beta, \sigma_\beta), \quad \text{for all } x \in \mathbb{R}, \tag{2.13}$$

where $f(x, \mu_\alpha, \sigma_\alpha)$ is the normal density function of random variable α with mean μ_α and standard deviation σ_α and $f(x, \mu_\beta, \sigma_\beta)$ is the normal density function of random variable β with mean μ_β and standard deviation σ_β.

In the following, we will use a 1D MND to calibrate and fit the empirical distribution of $y_{i1}, y_{i2}, \ldots, y_{im}$, where $i = 1, \ldots, n$, the log-return data of each transformed risk factor.

Now let us consider the following question. If we assume that the daily log-return data of one transformed risk factor has an MND, then how can we identify the parameters? Since each risk factor has a zero mean, after rotation each transformed risk factor must have zero mean too. In the following, we attach a sub-index i to a parameter to indicate that this parameter is corresponding to the ith transformed risk factor. For example, the variance of the transformed data of ith risk factor is denoted by σ_i^2. Thus, in (2.13), for each $i = 1, \ldots, n$ we have $\mu_{\alpha_i} = 0 = \mu_{\beta_i}$, where α_i and β_i are the two normal random variables in the MND which corresponds to the ith transformed risk factor. Consider the calibration of the second moment, fourth moment and sixth moment. By (2.13), we have the following system of calibration equations.

$$\begin{cases} p_i\sigma_{\alpha_i}^2 + (1-p_i)\sigma_{\beta_i}^2 = \sigma_i^2, \\ p_i m_{\alpha_i}^4 + (1-p_i)m_{\beta_i}^4 = \kappa_i(\sigma_i^2)^2, \\ p_i m_{\alpha_i}^6 + (1-p_i)m_{\beta_i}^6 = m_i^6, \end{cases} \tag{2.14}$$

where m_i^6 is the data sixth moment, κ_i is the data kurtosis, $\sigma_{\alpha_i}^2$, $m_{\alpha_i}^4$ and $m_{\alpha_i}^6$ are the second moment, fourth moment and sixth moment of α_i, respectively, and $\sigma_{\beta_i}^2$, $m_{\beta_i}^4$ and $m_{\beta_i}^6$ are the second moment, fourth moment and sixth moment of β_i, respectively. Since α_i and β_i have normal distributions, we get the following equivalent system of equations:

$$\begin{cases} p_i\sigma_{\alpha_i}^2 + (1-p_i)\sigma_{\beta_i}^2 = {\sigma_i}^2, \\ 3p_i(\sigma_{\alpha_i}^2)^2 + 3(1-p_i)(\sigma_{\beta_i}^2)^2 = \kappa_i({\sigma_i}^2)^2, \\ 15p_i(\sigma_{\alpha_i}^2)^3 + 15(1-p_i)(\sigma_{\beta_i}^2)^3 = m_i^6. \end{cases} \tag{2.15}$$

To simplify the notations, let $x = \sigma_{\alpha_i}^2$, $y = \sigma_{\beta_i}^2$, $a = {\sigma_i}^2$, $b = m_i^6/(15)$, $p = p_i$ and $c = \kappa_i({\sigma_i}^2)^2/3$. From (2.15) we can get

$$\begin{cases} px + (1-p)y = a, \\ px^2 + (1-p)y^2 = c, \\ px^3 + (1-p)y^3 = b. \end{cases} \tag{2.16}$$

Then, the solutions of (2.16) can be found as follows:

$$y_1 = \frac{-(b-ac) + \sqrt{(b-ac)^2 - 4(a^2-c)(c^2-ab)}}{2(a^2-c)} \tag{2.17}$$

and

$$y_2 = \frac{-(b-ac) - \sqrt{(b-ac)^2 - 4(a^2-c)(c^2-ab)}}{2(a^2-c)}. \tag{2.18}$$

$$x_i = \frac{c - a y_i}{a - y_i}, \quad i = 1, 2 \tag{2.19}$$

and

$$p_i = \frac{a - y_i}{x_i - y_i}, \quad i = 1, 2. \tag{2.20}$$

In the Appendix, we will discuss the properties of the solutions of the system of calibration equations. Thus, we have all the parameters $p_i, \sigma_{\alpha_i}, \sigma_{\beta_i}, i = 1, \dots, n$, and we can generate scenarios. For each integer i satisfying $1 \leqslant i \leqslant n$, let $\{e_{i1}, \dots, e_{ir}\}$ be the random numbers generated from a standard normal distribution and $\{e_{i1}, \dots, e_{ir}, 1 \leqslant i \leqslant n\}$ be independent. Let F_i be a 1D distribution function with density function:

$$p_i f(x, \mu_{\alpha_i}, \sigma_{\alpha_i}) + (1 - p_i) f(x, \mu_{\beta_i}, \sigma_{\beta_i}), \quad \text{for all } x \in \mathbb{R}.$$

Define $\tilde{y}_{ij} = F_i^{-1}(N(e_{ij})), i = 1, \dots, n, j = 1, \dots, r$ and

$$\tilde{\boldsymbol{Y}} = \begin{pmatrix} \tilde{y}_{11} & \tilde{y}_{12} & \cdots & \tilde{y}_{1r} \\ \tilde{y}_{21} & \tilde{y}_{22} & \cdots & \tilde{y}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{y}_{n1} & \tilde{y}_{n2} & \cdots & \tilde{y}_{nr} \end{pmatrix}. \tag{2.21}$$

Let

$$\tilde{\boldsymbol{X}} = \hat{\boldsymbol{U}} \tilde{\boldsymbol{Y}}. \tag{2.22}$$

By the law of large numbers, we have

$$\tilde{\boldsymbol{\Gamma}} \equiv \frac{1}{r} \tilde{\boldsymbol{Y}} \tilde{\boldsymbol{Y}}^\top \to \boldsymbol{\Lambda} \quad \text{a.s.}$$

as $r \to \infty$ and

$$\hat{\boldsymbol{U}}^\top \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{U}} = \frac{1}{m} \boldsymbol{Y} \boldsymbol{Y}^\top = \hat{\boldsymbol{\Gamma}} \to \boldsymbol{\Lambda} \quad \text{a.s.}$$

as $m \to \infty$. Therefore, we have

$$\frac{1}{r} \tilde{\boldsymbol{X}} \tilde{\boldsymbol{X}}^\top = \hat{\boldsymbol{U}} \frac{1}{r} \tilde{\boldsymbol{Y}} \tilde{\boldsymbol{Y}}^\top \hat{\boldsymbol{U}}^\top = \hat{\boldsymbol{U}} \tilde{\boldsymbol{\Gamma}} \hat{\boldsymbol{U}}^\top \doteq \hat{\boldsymbol{U}} \hat{\boldsymbol{\Gamma}} \hat{\boldsymbol{U}}^\top = \hat{\boldsymbol{\Sigma}} \tag{2.23}$$

as r and m are big enough, where $\doteq$ means approximate equality. Then, $\tilde{\boldsymbol{X}}$ is just the generated scenarios we desired.

From the generated scenarios of the risk factors, we can estimate the portfolio extreme VaR and VaR CI by non-parametric method. Now let us discuss this method.

3. Non-parametric method and portfolio VaR and VaR CI estimation

In this case, we follow the method of RiskMetrics (see Section 2.2 or page 18 on Monte Carlo simulation in Mina & Xiao, 2001). Suppose that we have h instruments, $V_j(\boldsymbol{R})$, $j = 1, \ldots, h$, in a portfolio, where the present value of each instrument is a function of n risk factors, i.e. $V_j(\boldsymbol{R})$, $j = 1, \ldots, h$ and $\boldsymbol{R} = (r_1, \ldots, r_n)$. We can obtain a 1-day profit and loss (P & L) scenario of the portfolio as follows:

1. Generate $\tilde{\boldsymbol{X}}$ as above. Pick up one column, say $(\tilde{x}_{1i}, \ldots, \tilde{x}_{ni})^\top$, where $1 \leqslant i \leqslant r$.
2. Set

$$r_{ki}(t+1) = r_{ki}(t)e^{\tilde{x}_{ki}} \quad \text{for } k = 1, \ldots, n.$$

3. Define $\boldsymbol{R}_i(t+1) = (r_{1i}(t+1), \ldots, r_{ni}(t+1))$. Then we can get the 1-day portfolio P & L

$$u_i = \sum_{j=1}^{h}(V_j(\boldsymbol{R}_i(t+1)) - V_j(\boldsymbol{R}_i(t))).$$

Thus, we have generated 1-day portfolio P & L scenarios $\{u_i : 1 \leqslant i \leqslant r\}$. Let $\tilde{F}_\eta(x)$ be the distribution of the 1-day portfolio P & L. For a given $\theta \in (0, 1)$, let q_θ be the θ quantile of $\tilde{F}_\eta$. Then,

$$\mathbb{P}(\eta \geqslant q_\theta) = 1 - \theta, \tag{3.1}$$

where η is a random variable that has distribution $\tilde{F}_\eta(x)$. In order to find the 1-day portfolio P & L VaR CI, define

$$\Lambda = \#\{i : u_i \leqslant q_\theta\}. \tag{3.2}$$

Then Λ has a binomial distribution $b(r, \theta)$ that has mean $r\theta$, standard deviation $\sqrt{r\theta(1-\theta)}$, and the distribution of $(\Lambda - r\theta)/\sqrt{r\theta(1-\theta)}$ converges to the standard normal distribution as r tends to infinity. That is

$$\mathbb{P}\left(-Z_{\alpha/2} < \frac{\Lambda - r\theta}{\sqrt{r\theta(1-\theta)}} < Z_{\alpha/2}\right) \approx 1 - \alpha \tag{3.3}$$

for r big enough. After a little adjustment, we have

$$\mathbb{P}(\pi \leqslant \Lambda \leqslant \Pi) \geqslant 1 - \alpha \quad \text{for } r \text{ big enough,} \tag{3.4}$$

where

$$\pi = [[r\theta - (Z_{\alpha/2} + 0.05)\sqrt{r\theta(1-\theta)}]], \tag{3.5}$$

$$\Pi = [[r\theta + (Z_{\alpha/2} + 0.05)\sqrt{r\theta(1-\theta)}]] + 1, \tag{3.6}$$

and $[[a]]$ is the maximum integer less than a.

Let $Y^1 \leqslant Y^2 \leqslant \cdots \leqslant Y^r$ be the order statistics constructed from the scenarios $u_1, \ldots, u_r$. Then the above inequality (3.4) is equivalent to

$$\mathbb{P}(Y^\pi \leqslant q_\theta \leqslant Y^\Pi) \geqslant 1 - \alpha, \quad \text{for } r \text{ big enough.} \tag{3.7}$$

This gives the non-parametric VaR CI for the 1-day portfolio P & L. To find the VaR, define

$$\aleph = \min\{i : 0 \leqslant i \leqslant r, Y^i \geqslant q_\theta\}. \tag{3.8}$$

Then, $Y^\aleph$ is the estimated VaR of the 1-day portfolio P & L with probability level θ.

4. Model validation and data analysis

In this section, we will use the quantile–quantile plot (qq-plot) method, graph match, covariance matrix match and quantile comparison to test our model. Although we have tested different data sets and different risk factors, here we only briefly calibrate one data set in this study. The data set consists of five risk factors of 2720 observations of daily log-returns of USD/AUD (US Dollar/Australia Dollar), USD/CAD (US Dollar/Canadian Dollar), USD/CHF (US Dollar/Liechtenstein Franc), USD/GBP (US Dollar/British Pound) and USD/JPY(US Dollar/Japanese Yen) exchange rates, from 8 August 1989 to 20 January 2000. We use our model to calibrate the desired parameters. First, we use the qq-plot method to check the 1D marginal distribution match. From the Glivenko-Cantelli Theorem (cf. Example 2.1.4. of Embrechts, 1997), if the MND with found parameters fits well the empirical distribution of the 1D data, it must demonstrate approximate linearity of the qq-plot. In order to demonstrate the degree of the calibrated MND fitting to the market data and also avoid repetitiveness, we will only give one qq-plot. The plot is the marginal qq-plot for the scenarios generated by the MND with calibrated parameters for one risk factor versus the market log-return data of the corresponding risk factor. To get a clear view of the density curves, we illustrate and compare our model density curve, the found density curve of the MND, with the data histogram and a normal density curve which has mean and variance the same as that of the data histogram for one risk factor. Finally, we consider the estimation of the (extreme) VaR and VaR CI of a special portfolio which is the sum of five instruments, where each instrument is just 100 shares of one exchange rate measured in US dollars. We assume that today's exchange rates are 1.5034 for USD/AUD, 1.4438 for USD/CAD, 1.5975 for USD/CHF, 0.6075 for USD/GBP and 105.47 for USD/JPY. We use tables to compare the VaRs and VaR CIs of the portfolio P & L scenarios generated by the normal model and the model of an MND with different confidence levels. This demonstrates that the mixture of normal models is more effective for handling the heavy-tailed cases than the normal model for our constructed portfolio.

The parameters of the histogram and the different density curves in the Fig. 2 are listed in Table 1 and Table 2.

Then, we use our model to generate scenarios. We illustrate the scenario histogram and compare the model density curve of the MND to the histogram of the generated scenarios and to the normal density curve with the same mean and variance as that of the data histogram.

The parameters of the histogram of the 10000 scenarios in Fig. 3 are listed in Table 3.

As for the calibration of covariance matrix, we have the following covariance matrix of the market daily log-return data.

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.3066 & 0.0156 & 0.0119 & 0.0332 & 0.0141 \\ 0.0156 & 0.0843 & -0.0121 & -0.0071 & -0.0029 \\ 0.0119 & -0.0121 & 0.5491 & 0.3116 & -0.0057 \\ 0.0332 & -0.0071 & 0.3116 & 0.3846 & 0.0013 \\ 0.0141 & -0.0029 & -0.0057 & 0.0013 & 0.5612 \end{pmatrix}. \tag{4.1}$$

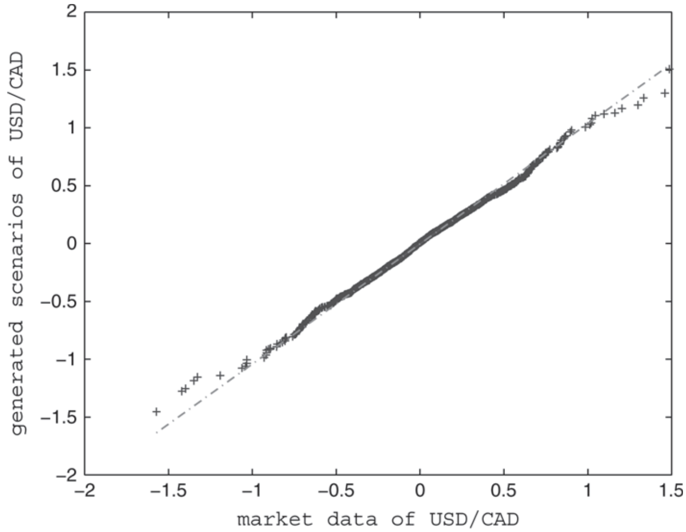

FIG. 1. qq-plot for the market daily log-return data of exchange rates of USD/CAD versus the scenarios generated by the MND with calibrated parameters. It demonstrates approximate linearity.

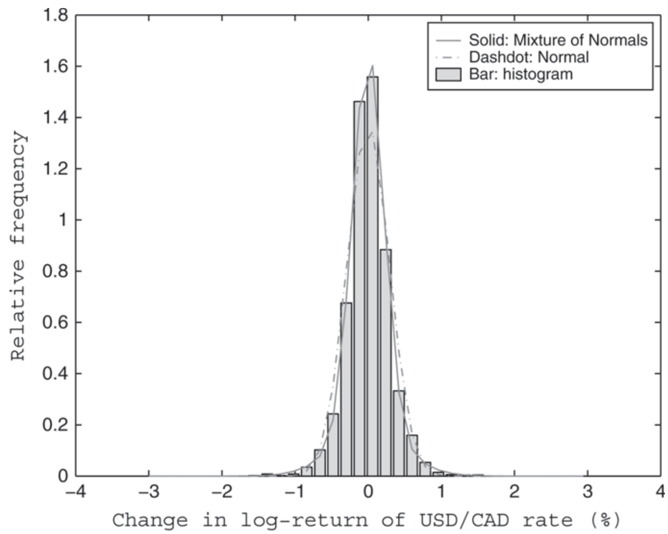

FIG. 2. A comparison of the histogram of 2720 historical observations of daily log-return of USD/CAD exchange rates with the calibrated density curve of an MND, and the normal density curve with the same mean and variance as that of the histogram.

We have the following covariance matrix of the scenarios which are generated from the calibrated MND:

$$\hat{\hat{\Sigma}} = \begin{pmatrix} 0.2983 & 0.0150 & 0.0133 & 0.0323 & 0.0144 \\ 0.0150 & 0.0843 & -0.0121 & -0.0072 & -0.0028 \\ 0.0133 & -0.0121 & 0.5511 & 0.3098 & -0.0058 \\ 0.0323 & -0.0072 & 0.3098 & 0.3877 & 0.0012 \\ 0.0144 & -0.0028 & -0.0058 & 0.0012 & 0.5587 \end{pmatrix}. \tag{4.2}$$

TABLE 1 *Parameters of histogram and the normal curve*

	Mean	Variance	Kurtosis	Fourth moment	Sixth moment
Histogram	0.0	0.0843	5.5664	0.0395	0.0435
Normal	0.0	0.0843	3	0.0213	0.009

TABLE 2 *Parameters of the density curve of the MND*

	μ_{α_1}	μ_{β_1}	σ_{α_1}	σ_{β_1}	p
Mixture of normals	0.0	0.0	0.497	0.216	0.8143

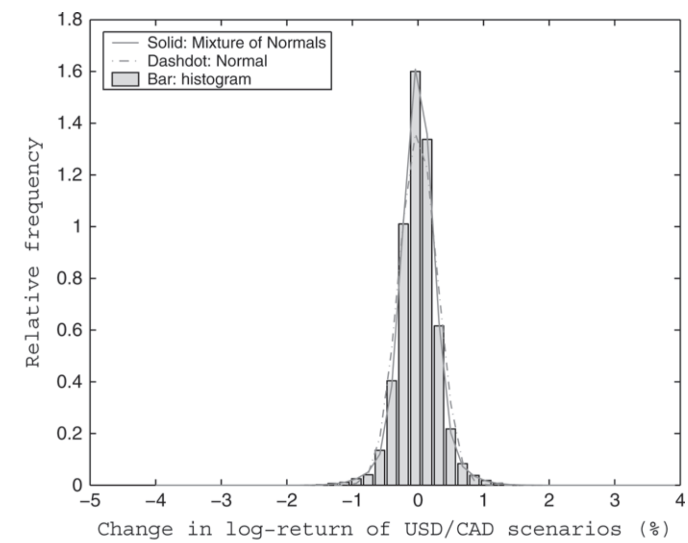

FIG. 3. A comparison of the histogram of 10000 scenarios generated by the calibrated MND with the calibrated density curve of the MND and the normal density curve with the same mean and variance as that of the histogram.

TABLE 3 *Parameters of the histogram of the 10000 scenarios*

Mean	Variance	Kurtosis	Fourth moment	Sixth moment
0.0000	0.0834	5.528	0.039	0.0433

Their difference is as follows:

$$\hat{\Sigma} - \hat{\hat{\Sigma}} = \begin{pmatrix} 0.0082 & 0.0006 & -0.0013 & 0.0009 & -0.0003 \\ 0.0006 & 0.0000 & -0.0001 & 0.0000 & -0.0000 \\ -0.0013 & -0.0001 & -0.0020 & 0.0018 & 0.0000 \\ 0.0009 & 0.0000 & 0.0018 & -0.0031 & 0.0000 \\ -0.0003 & -0.0000 & 0.0000 & 0.0000 & 0.0025 \end{pmatrix}. \tag{4.3}$$

Finally, we validate our model by a special portfolio which is the sum of five instruments, where each instrument is 100 shares of the value of one US dollar measured in one foreign currency, respectively. Since the Black-Scholes formula does not work in the heavy-tailed case, we have not included option instruments in the portfolio. From the generated scenarios of the risk factors, we find the histogram of the P & L scenarios of the portfolio as follows.

The parameters of the normal density curve and the histogram of the 10000 P & L scenarios of the portfolio in Fig. 4 are listed in Table 4.

Now we use tables to compare the VaRs and VaR CIs of the histograms of the normal scenarios and the portfolio P & L scenarios derived from five risk factors modelled and simulated by the MND.

To conclude, we give some comments on the simulation results. First, the illustration demonstrates the well fitting of the calibrated density curve of the MND to the histogram of the data of 2720 historical

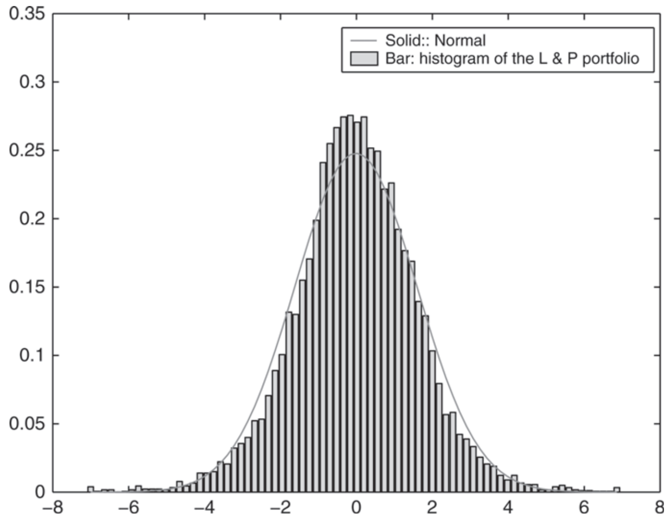

FIG. 4. A comparison of the histogram of 10000 P & L scenarios of the portfolio with the normal density curve having the same mean and variance as that of the histogram.

TABLE 4 *Parameters of the histogram of the 10000 scenarios and the normal density curve*

	Mean	Variance	Kurtosis
Histogram of the scenarios	−0.048	2.5575	4.2992
Normal density	−0.048	2.5575	3

TABLE 5 *VaR(α) means VaR with confidence level α(100%). VaRCILB(α) means the 95% VaR CI lower bound of the VaR with confidence level α(100%) and VaRCIUB(α) means the 95% VaR CI upper bound of the VaR with confidence level α(100%)*

	Size	VaR(0.05)	VaRCILB(0.05)	VaRCIUB(0.05)
Normal	10000	2.6395	2.6064	2.7305
Scen	10000	2.6489	2.4974	2.6635
	Size	VaR(0.01)	VaRCILB(0.01)	VaRCIUB(0.01)
Normal	10000	3.7212	3.6271	3.807
Scen	10000	4.3196	4.0986	4.5732
	Size	VaR(0.001)	VaRCILB(0.001)	VaRCIUB(0.001)
Normal	10000	5.1332	4.5995	5.6457
Scen	10000	6.3751	5.7902	7.7826

observations of daily log-returns of the risk factor. Second, by a rotation method we effectively calibrated the covariance matrix. Third, as of the estimation of VaR and VaR CI of the P & L portfolio values, from the comparison table we can see that the normal model has seriously underestimated the extreme VaR and VaR CI. From the above illustration and validation, we can see that our scenario generation method combining with the non-parametric estimation give us a powerful tool for the estimation of the extreme VaR and VaR CI of heavy-tailed risk factors.

Acknowledgements

The authors would like to take this opportunity to express their gratitude to the referee, Professor Reg J. Kulperger and Professor David A. Levin for their careful reading of the paper, their comments and helpful suggestions. In particular, the elegant short proof of Theorem A.2 is due to the referee.

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Appendix. Analysis of the solutions of the calibration equations

We have found the closed-form solutions of (2.16), the system of calibration equations. However, one may have the following question about the solutions: Do the solutions given by (2.20) satisfy $0 \leqslant p_i \leqslant 1$? Since (2.15) and (2.16) are equivalent and the solutions of (2.15) should be non-negative, are the solutions given by (2.17) and (2.18) non-negative? We answer here these questions. In order to simplify the notation, in the following discussion, we drop the sub-index i. Recall that we have $a = \sigma^2$, $b = m^6/(15)$ and $c = \kappa(\sigma^2)^3/3$.

LEMMA A.1 Let p, given by (2.20), be a solution of (2.16). Then, $0 \leqslant p \leqslant 1$.

Proof. First, for any heavy-tailed distribution the kurtosis $\kappa > 3$. Thus, $a^2 - c = (\sigma^2)^2 - \frac{\kappa(\sigma^2)^2}{3} < 0$. If $a < y$, by (2.19) we have

$$a - x = \frac{a^2 - ay}{a - y} - \frac{c - ay}{a - y} = \frac{a^2 - c}{a - y} > 0. \tag{A.1}$$

Since $y > a > x$, we get

$$0 \leqslant p = \frac{a - y}{x - y} \leqslant 1.$$

On the other hand, if $a > y$, by (2.19) we have

$$x - a = \frac{c - ay}{a - y} - \frac{a^2 - ay}{a - y} = \frac{c - a^2}{a - y} > 0. \tag{A.2}$$

Since $x > a > y$, we get

$$0 \leqslant p = \frac{a-y}{x-y} \leqslant 1.$$

□

THEOREM A.2 The solutions of (2.16), given by (2.17), (2.18) and (2.19), are non-negative.

Proof. To prove that $y_1 > 0$ and $y_2 > 0$, since $a^2 < c$, we only need to prove that $b - ac > 0$ and $c^2 - ab < 0$. For given $0 < p < 1$. Let Z be a random variable defined by

$$Z = \begin{cases} x, & \text{with probability } p, \\ y, & \text{with probability } 1-p, \end{cases}$$

where $x = \sigma_\alpha^2 > 0$ and $y = \sigma_\beta^2 > 0$. Then, $\mathbb{E}Z = a$, $\mathbb{E}Z^2 = c$ and $\mathbb{E}Z^3 = b$.

(i) $\mathrm{var}(Z) = \mathbb{E}Z^2 - (\mathbb{E}Z)^2 = c - a^2 > 0$.

(ii) Clearly, $\mathrm{cov}(Z, Z^2) > 0$ for positive random variable Z, so

$$\mathrm{cov}(Z, Z^2) = \mathbb{E}Z^3 - (\mathbb{E}Z)(\mathbb{E}Z^2) = b - ac > 0.$$

(iii) By Cauchy–Schwarz's inequality, we have

$$(\mathbb{E}Z^2)^2 = \{\mathbb{E}(Z^{1/2}Z^{3/2})\}^2 < (\mathbb{E}Z)(\mathbb{E}Z^3).$$

This gives $c^2 - ab < 0$. Now from the symmetric property of (2.16), with same argument as before we can get the non-negativity of x_1 and x_2. This completes the proof of the theorem. □