Acta Mathematica Sinica, English Series
Sep., 201x, Vol. x, No. x, pp. 1–21
Published online: August 15, 201x
DOI: 0000000000000000
Http://www.ActaMath.com

Conditional log-Laplace Functional for a Class of Branching Processes in Random Environments

Hao WANG
Department of Mathematics, The University of Oregon,
Eugene, Oregon 97403-1222, U.S.A.
E-mail: *haowang@uoregon.edu*

Abstract A conditional log-Laplace functional (CLLF) for a class of branching processes in random environments is derived. The basic idea is the decomposition of a dependent branching dynamic into a no-interacting branching and an interacting dynamic generated by the random environments. CLLF will play an important role in the investigation of branching processes and superprocesses with interaction.

Keywords interacting superprocess, conditional log-Laplace functional, branching process in random environment, Wong-Zakai approximation, duality.

MR(2010) Subject Classification 60J68, 60J80, 60K35

1 Introduction

In this paper, in order to present ideas and method clearly we will introduce a simplified model for a class of underlying particle motion-free superprocesses in random environments (SDBMs). However, the models with motion, even with interacting motion cases can be handled similarly as long as we can correctly derive and handle the corresponding stochastic evolution equation for the dual processes. The SDBMs can be constructed by the martingale problem method. As usual, the uniqueness of the martingale problem for SDBMs gives us a challenge because the random environments produce interaction dynamic in the branching mechanism. Due to the loss of infinite divisibility in our model, lot of existing methods are not available for the model. It is natural to turn to other tools or methods which can handle the model such as duality method. After checking Dawson-Kurtz's duality method ([4]), we found that our model is a non-conservative case. Thus it requires that the moment generating function for the associated branching processes of the SDBMs has a positive radius of convergence. However, this may not be true for our model. Therefore, Dawson-Kurtz's duality method does not directly work for our model. Since SDBMs have a dependent branching mechanism, they do not have infinite divisibility. So the moment generating functional duality and the log-Laplace functional duality can not be derived based on infinite divisibility. Nevertheless, they do have conditional infinite divisibility given random environments. Based on this fact and understanding, we will derive a stochastic evolution equation for the dual process and its Wong-Zakai approximation equation. Then, the duality argument and the Wong-Zakai approximation solution jointly yield

Received December 24, 2013, accepted x x, 2014

the dual identity or the conditional log-Laplace functional which will serve as a basic tool for the construction and the investigation of properties of the branching processes and superprocesses with interacting branching mechanism.

Branching processes in random environments (BPREs) were introduced by Smith and Wilkinson [18]. The diffusion approximation of BPREs was conjectured by Keiding [10] and constructed by Kurtz [11] as the unique solution $(Z_t, S_t)_{t\geq 0}$ of the following system of two stochastic differential equations:

$$\begin{aligned} dZ_t &= \tfrac{1}{2}\sigma_e^2 Z_t dt + Z_t dS_t + \sqrt{\sigma_b^2 Z_t} dW_t^{(b)} \\ dS_t &= \alpha dt + \sqrt{\sigma_e^2} dW_t^{(e)} \end{aligned} \tag{1.1}$$

where $t \geq 0, Z_0 = z, S_0 = 0$, $W_t^{(b)}$ and $W_t^{(e)}$ are two independent standard Brownian motions. Recently along this line Böinghoff-Hutzenthaler [1] and Hutzenthaler [8] have made further interesting development. In [1] and [8], the infinitesimal drift α of the associated Brownian motion $(S_t)_{t\geq 0}$ is used to determine the type of criticality, the Laplace transformation of the Feller branching diffusion is used to derive the Laplace transformation of the BDRE conditioned on the associated Brownian motion and discovered the phase transition in supercritical regime and the characterization of the survival probability of $(Z_t)_{t\geq 0}$ according different regimes. Here we only name a few references. The reader is referred to Böinghoff-Hutzenthaler [1] and Hutzenthaler [8] for more details and Mytnik [15] for the related space-time model and references therein. For branching processes and superprocesses, the reader is referred to Dawson [3], Perkins [16], Li [12], and Xiong [22].

In this paper, we view branching diffusion in the random environments as a motion-free superprocess in the random environments. Then, we can use dual stochastic evolution equation and duality method to construct the function-valued dual processes and the conditional log-Laplace functional. The more interesting part is that this methodology can be generalized to handle branching particle systems with interacting motion and dependent branching mechanism. The present paper was motivated from the consideration to generalize a model studied in Wang [20], [19] and Dawson et al. [5] by introducing dependent branching mechanism. The construction of the conditional log-Laplace functional duality for interacting superprocesses given by Li et al. [13] and Xiong [21] has inspired the author for the current work.

The paper is organized as follows. In the first section, we introduce our model with interacting branching, describe the difficulties and challenges we encounter and how we overcome them. Then we will give a construction of interacting branching particle systems, establish the existence of the SDBMs. In the second section, we derive the dual stochastic evolution equation for the dual processes of the branching processes. In order to construct the conditional log-Laplace functional duality, we introduce a Wong-Zakai type approximation equation of the dual stochastic evolution equation. Then, we will prove that both the dual stochastic evolution equation and its Wong-Zakai type approximation equation have unique solutions and also the solution of the Wong-Zakai type approximation equation converges to the solution of the dual stochastic evolution equation in $L^2(\mathbb{P})$ sense. In section 3, we introduce conditional martingale problem with Wong-Zakai type approximation and discuss the convergence of the approximation solutions. In the last section, using the results of the previous sections and the dual argument, we will derive the conditional log-Laplace functional duality and we will discuss the application of the conditional log-Laplace functional duality and prove that the martingale problem for $\mathcal{B}$ discussed in the first section has uniqueness.

1.1 Model Description

Now let us describe the model we are interested in. First, in order to give an identity for each particle in a branching particle system, we need to introduce an index set to identify each particle in the branching tree structure. Let $\Re$ be the set of all multi-indices, i.e., strings of the form $\alpha = n_1 \oplus n_2 \oplus \cdots \oplus n_k$, where the n_i's are non-negative integers. Let $|\alpha|$ denote the length of α. We provide $\Re$ with the arboreal ordering: $m_1 \oplus m_2 \oplus \cdots \oplus m_p \prec n_1 \oplus n_2 \oplus \cdots \oplus n_q$ if and only if $p \leq q$ and $m_1 = n_1, \cdots, m_p = n_p$. If $|\alpha| = p$, then α has exactly $p-1$ predecessors, which we shall denote respectively by $\alpha - 1$, $\alpha - 2, \cdots, \alpha - |\alpha| + 1$. For example, with $\alpha = 6 \oplus 18 \oplus 7 \oplus 9$, we get $\alpha - 1 = 6 \oplus 18 \oplus 7$, $\alpha - 2 = 6 \oplus 18$ and $\alpha - 3 = 6$. We also define an $\oplus$ operation on $\Re$ as follows: if $\eta \in \Re$ and $|\eta| = m$, for any given non-negative integer k, $\eta \oplus k \in \Re$ and $\eta \oplus k$ is an index for a particle in the $(m+1)$-th generation. For example, when $\eta = 3 \oplus 8 \oplus 17 \oplus 2$ and $k = 1$, we have $\eta \oplus k = 3 \oplus 8 \oplus 17 \oplus 2 \oplus 1$.

Let $\{B_t, t \geq 0\}$ and $\{W_t, t \geq 0\}$ be two independent standard $\mathbb{R}$-valued Brownian motions. Assume that W and B are defined on a common right continuous filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and independent of each other.

For a positive integer k, let $L^k(\mathbb{R}^d)$ be the Banach space of functions on $\mathbb{R}^d$ satisfying $\|f\|_k < \infty$, where the norm

$$\|f\|_k := \{\int_{\mathbb{R}^d} |f(x)|^k dx\}^{1/k}.$$

Let $L^\infty(\mathbb{R}^d)$ be the space of bounded Lebesgue measurable functions on $\mathbb{R}^d$ with norm $\|\cdot\|_\infty$. Denote by $C((\mathbb{R}^d)^m)$ and $C^k((\mathbb{R}^d)^m)$ the space of continuous functions on $(\mathbb{R}^d)^m$ and the space of continuous functions on $(\mathbb{R}^d)^m$ with continuous derivatives up to and including order k, respectively.

We assume that each $\mathbb{R}$-valued particle is spatial motion-free.

1.2 Branching Particle Systems

We now consider the branching particle systems in which each particle is spatial motion-free. For every positive integer $n \geq 1$, there is an initial system of $m_0^{(n)}$ particles. Each particle has mass $1/\theta^n$ and branches at steps $\{k/(\gamma\theta^n), k \in \mathbb{N}\}$, where $\gamma > 0$ and $\theta > 1$ are fixed constants. In order to simplify notation, in the following we set $\gamma = 1$. In the following we introduce a dependent branching mechanism in our model. Let $\xi(t) : [0, \infty) \to \mathbb{R}$ be a mean-zero random process. We assume that for any $0 \leq s < t < \infty$, $\xi(s)$ and $\xi(t)$ are independent and $\xi(t)$ is uniformly bounded. In other word, there is a constant $M > 0$ and a common zero probability set N such that $\sup_{t\in[0,\infty)} |\xi(t, \omega)| \leq M$ holds for each $\omega \notin N$. Without loss of generality, we may assume that $M = 1$. Thus, for any $0 \leq s \leq t < \infty$, the covariance function

$$\mathbb{E}\xi(s)\xi(t) = \delta_s(t)g^2,$$

where $\delta_s(t)$ is equal to 1 or 0 according as $t = s$ or $t \neq s$ and g is a constant satisfying $0 \leq g \leq 1$. For any $t \in [0, \infty)$, define

$$q_0^{(n)}(t) := \frac{a_0}{2} - \frac{b_0 \xi(t)}{2\sqrt{\theta^n}}, \qquad q_1^{(n)}(t) := 0,$$

and

$$q_k^{(n)}(t) := \frac{a_k}{2^k} + \frac{b_k \xi(t)}{2^k \sqrt{\theta^n}}, \qquad k \geq 2,$$

where for any integer $k \geq 0$, we assume that the constants a_k and b_k satisfy $2^{k-1} \geq a_k \geq b_k \geq 0$. Then, we have $q_k^{(n)}(t) \geq 0$. In order to present the main idea clearly, in the following we simplify our notation and assume that $a_0 = 1$, $b_0 = 1$, $a_2 = 2$, $b_2 = 2$ and $a_k = 0$, $b_k = 0$ for all $k \geq 3$. Then we have

$$\sum_{k=0}^{\infty} q_k^{(n)}(t) \equiv 1,$$

$$0 \leq M_1^{(n)}(t) := \sum_{k=0}^{\infty} k q_k^{(n)}(t) = 1 + \frac{\xi(t)}{\sqrt{\theta^n}} \leq 1 + \frac{1}{\sqrt{\theta^n}}.$$

Let

$$M_2^{(n)}(t) := \sum_{k=0}^{\infty} k^2 q_k^{(n)}(t).$$

Then,

$$0 \leq \sup_{t \in [0,\infty)} M_2^{(n)}(t) \leq 4.$$

We assume that at the n^{th} stage when a particle dies at time t, it produces $k \geq 0$ offspring with probability $q_k^{(n)}(t)$. We assume that $m_0^{(n)} \leq \hbar\, \theta^n$, where $\hbar > 0$ and $\theta > 1$ are fixed constants. For a fixed $n \geq 1$, let $\Xi_n := \{\frac{k}{\theta^n} : k \in \mathbb{N}\}$ and let $\alpha \in \Re$, $\{O_\alpha^{(n)}(t) : \alpha \in \Re\}$ be a family of random processes, which are conditional independent to each other given $\xi(\cdot)$, such that for any $t \in ([0, \infty) \setminus \Xi_n)$, $\mathbb{P}(O_\alpha^{(n)}(t) = 1) = 1$ and $k = 0$ or 2,

$$\mathbb{P}(O_\alpha^{(n)}(t) = k | \xi(t)) = \begin{cases} q_k^{(n)}(t), & \text{if } t \in \Xi_n, \\ 0, & \text{if } t \notin \Xi_n. \end{cases} \tag{1.2}$$

In this interacting branching model, we simply take $\{C_\alpha^{(n)} = \frac{1}{\theta^n} : \alpha \in \Re\}$, (Note: in the usual independent branching model, $\{C_\alpha^{(n)} : \alpha \in \Re\}$ is assumed as a family of i.i.d. real-valued exponential random variables with parameter θ^n), which will serve as lifetimes of the particles. We assume W and B are independent and they are independent of $\{O_\alpha^{(n)} : \alpha \in \Re\}$. In our model, once the particle α dies, it disappears from the system.

In the remainder of this section we are only concerned with stage n. To simplify our notation, we will use the convention of dropping the superscript (n) from the random variables. In later sections we will continue this convention for some random variables such as birth times and death times. This will not cause any confusion, since the stage should be clear from the context. If $\beta(\alpha - 1)$ is the birth time of the particle $\alpha - 1$, then the birth time $\beta(\alpha)$ of the particle α is defined backward in time, recursively by

$$\beta(\alpha) := \begin{cases} \beta(\alpha-1) + C_{\alpha-1} = \frac{|\alpha|-1}{\theta^n}, & \text{if } O_{\alpha-1}(\zeta(\alpha-1)) = 2\ ; \\ \infty, & \text{otherwise.} \end{cases}$$

where $\zeta(\alpha-1)$ is the death time of the particle $\alpha-1$ which is defined by

$$\zeta(\alpha-1) = \beta(\alpha-1) + C_{\alpha-1}$$

and the indicator function of the lifespan of α is denoted by $\ell_\alpha(t) := 1_{[\beta(\alpha),\zeta(\alpha))}(t)$.

Let ∂ denote the cemetery point. Define $x_\alpha(t) = \partial$ if either $t < \beta(\alpha)$ or $t \geq \zeta(\alpha)$. We make a convention that any function f defined on $\mathbb{R}$ is automatically extended to $\mathbb{R} \cup \{\partial\}$ by setting $f(\partial) = 0$ — this allows us to keep track of only those particles that are alive at any given time.

To avoid the trivial case, we assume that $\mu_0 \in M_F(\mathbb{R})$, where $M_F(\mathbb{R})$ is the Polish space of all finite measures on $\mathbb{R}$ with weak convergence topology. Let $\mu_0^{(n)} := (1/\theta^n)\sum_{\alpha=1}^{m_0^{(n)}} \delta_{x_\alpha(0)}$ be constructed such that $\mu_0^{(n)} \Rightarrow \mu_0$ as $n \to \infty$. We are thus provided with a collection of initial starting points $\{x_\alpha(0)\}$ for each $n \geq 1$.

Let $\mathcal{N}_1^n := \{1, 2, \cdots, m_0^{(n)}\}$ be the set of indices for the first generation of particles. For any $\alpha \in \mathcal{N}_1^n \cap \Re$, define

$$x_\alpha(t) := \begin{cases} x_\alpha(0), & t \in [0, C_\alpha), \\ \partial, & t \geq C_\alpha, \end{cases} \tag{1.3}$$

and

$$x_\alpha(t) \equiv \partial \qquad \text{for any } \alpha \in (\mathbb{N} \setminus \mathcal{N}_1^n) \cap \Re \text{ and } t \geq 0.$$

If $\alpha_0 \in \mathcal{N}_1^n \cap \Re$ and $O_{\alpha_0}(\zeta(\alpha_0), \omega) = 2$, define for every $\alpha \in \{\alpha_0 \oplus i :\ i = 1, 2\}$,

$$x_\alpha(t) := \begin{cases} x_{\alpha_0}(\zeta(\alpha_0)-), & t \in [\beta(\alpha), \zeta(\alpha)), \\ \partial, & t \geq \zeta(\alpha) \text{ or } t < \beta(\alpha). \end{cases} \tag{1.4}$$

If $O_{\alpha_0}(\zeta(\alpha_0), \omega) = 0$, define $x_\alpha(t) \equiv \partial$ for $0 \leq t < \infty$ and $\alpha \in \{\alpha_0 \oplus i : i \geq 1\}$.

More generally for any integer $m \geq 1$, let $\mathcal{N}_m^n \subset \Re$ be the set of all indices for the particles in the m-th generation. If $\alpha_0 \in \mathcal{N}_m^n$ and if $O_{\alpha_0}(\zeta(\alpha_0), \omega) = 2$, define for $\alpha \in \{\alpha_0 \oplus i : i = 1, 2\}$

$$x_\alpha(t) := \begin{cases} x_{\alpha_0}(\zeta(\alpha_0)-), & t \in [\beta(\alpha), \zeta(\alpha)), \\ \partial, & t \geq \zeta(\alpha) \text{ or } t < \beta(\alpha). \end{cases} \tag{1.5}$$

If $O_{\alpha_0}(\zeta(\alpha_0), \omega) = 0$, define

$$x_\alpha(t) \equiv \partial \qquad \text{for } 0 \leq t < \infty \text{ and for } \alpha \in \{\alpha_0 \oplus i : i \geq 1\}.$$

Continuing in this way, we obtain a branching tree of particles for any given ω with random initial state taking values in $\left\{x_1(0), x_2(0), \cdots, x_{m_0^{(n)}}(0)\right\}$. This gives us our branching particle systems in $\mathbb{R} \cup \partial$, where particles undergo a finite-variance branching at time $t \in \Xi_n$ and have no spatial motion.

1.3 Tightness and Construction of SDBMs

Recall that $\{x_\alpha\}$ is the branching particle system constructed in the last section. Define its associated empirical process by

$$\mu_t^{(n)}(A) := \frac{1}{\theta^n}\sum_{\alpha\in\Re}\delta_{x_\alpha(t)}(A) \qquad \text{for } A\in\mathcal{B}(\mathbb{R}), \tag{1.6}$$

where $\mathcal{B}(\mathbb{R})$ denotes the family of Borel subsets of $\mathbb{R}$. In the following, we will show that $\{\mu_t^{(n)} : t\geq 0\}$ converges weakly as $n\to\infty$ and its weak limit is the SDBM on $\mathbb{R}$.

By Itô's formula, formally we can derive the infinitesimal generator of SDBM as follows.

$$\begin{aligned}\mathcal{B}F(\mu) &:= \frac{1}{2}\int_{\mathbb{R}}\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx)\\ &+ \frac{g^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\right)\mu(dx)\mu(dy),\end{aligned} \tag{1.7}$$

for $F(\mu)\in\mathcal{D}(\mathcal{B})\subset C(M_F(\mathbb{R}))$. Here $C(M_F(\mathbb{R}))$ is the space of all continuous functions on $M_F(\mathbb{R})$. The variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{h\downarrow 0}\frac{F(\mu+h\delta_x)-F(\mu)}{h}; \tag{1.8}$$

and $\mathcal{D}(\mathcal{B})$, the domain of the pregenerator $\mathcal{B}$, consists of functions of the form

$$F(\mu) = f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_k,\mu\rangle),$$

with $\phi_i\in C_c^2(\mathbb{R})$, $f\in C_b^2(\mathbb{R}^k)$, $k\in\mathbb{N}$. In the following we will construct the SDBM as unique solution to the martingale problem for $\mathcal{B}$. The proof of uniqueness of the martingale problem for $\mathcal{B}$ will be given in Section 4.

For any $t>0$ and $A\in\mathcal{B}(\mathbb{R})$, define

$$M^{(n)}(A\times(0,t]) := \sum_{\alpha\in\Re}\frac{[O_\alpha^{(n)}(\zeta(\alpha))-1]}{\theta^n}1_{\{x_\alpha(\zeta(\alpha)-)\in A,\,\zeta(\alpha)\leq t\}}. \tag{1.9}$$

$M^{(n)}(A\times(0,t])$ describes the space-time related branching in the set A up to time t. We will prove that $M^{(n)}$ is a martingale measure.

In order to use Mitoma's theorem (see [14]) to discuss the weak convergence of our empirical measure-valued processes, we introduce some new notation.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing test functions and $\mathcal{S}'(\mathbb{R})$ be the dual space of $\mathcal{S}(\mathbb{R})$, the space of Schwartz tempered distributions. Mitoma's theorem ([14]) provides a convenient tool for studying the weak convergence of measure-valued processes. It is applicable to càdlàg processes whose state space is the dual of a nuclear Fréchet space. A typical case is the $\mathcal{S}'(\mathbb{R})$-valued processes.

Note that for every $\phi\in\mathcal{S}(\mathbb{R})$, we have the following equation for our branching particle system:

$$\langle\phi,\mu_t^{(n)}\rangle - \langle\phi,\mu_0^{(n)}\rangle = M_t^{(n)}(\phi), \tag{1.10}$$

where, recall that $\ell_\alpha(s) = 1_{[\beta(\alpha),\,\zeta(\alpha))}(s)$,

$$M_t^{(n)}(\phi) := \int_0^t\int_{\mathbb{R}}\phi(x)M^{(n)}(dx,ds)$$

$$= \sum_{\alpha\in\Re} \frac{[O^{(n)}_\alpha(\zeta(\alpha)) - 1]}{\theta^n} \phi(x_\alpha(\zeta(\alpha)-))\, 1_{\{\zeta(\alpha)\le t\}}$$

Using a result of Dynkin ([6] p.325, Theorem 10.25), we immediately get the following theorem.

Theorem 1.1 *Let $\{C^{(n)}_\alpha\}$ be defined as before. For any $n \in \mathbb{N}$, $\mu^{(n)}_t$ defined by (1.6) is a right continuous strong Markov process which is the unique strong solution of (1.10) in the sense that it is a unique solution of (1.10) for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and given ξ, $\{O^{(n)}_\alpha\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, $\{\mu^{(n)}_t : t \ge 0, n \in \mathbb{N}\}$ are all defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof The existence and the uniqueness follow from the above construction of the branching particle systems. □

For each $t \ge 0$, let $\mathcal{F}^{(n)}_t$ denote the σ-algebra generated by the collection of processes

$$\left\{\mu^{(n)}_s(\phi), M^{(n)}_s(\phi) :\, \phi \in \mathcal{S}(\mathbb{R}),\, 0 \le s \le t\right\}.$$

Lemma 1.2 *With the notation above, we have the following.*

(i) *For every $\phi \in \mathcal{S}(\mathbb{R})$, $M^{(n)}(\phi) := \{M^{(n)}_t(\phi) : t \ge 0\}$ is a purely discontinuous square integrable martingale with*

$$\begin{aligned}
\langle M^{(n)}(\phi)\rangle_t &= \sum_{s_n\le t} \langle \phi^2, \mu^{(n)}_{s_n}\rangle \frac{1}{\theta^n} \\
&\quad + g^2 \sum_{s_n\le t} \langle \phi(x)\phi(y), \mu^{(n)}_{s_n}(dx)\mu^{(n)}_{s_n}(dy)\rangle \frac{1}{\theta^n} \\
&\quad - \frac{g^2}{\theta^n} \sum_{s_n\le t} \langle \phi^2(x), \mu^{(n)}_{s_n}(dx)\rangle \frac{1}{\theta^n} \qquad \textit{for every } t \ge 0 \\
&= \int_0^t \langle \phi^2, \mu^{(n)}_{s_n}\rangle ds \\
&\quad + g^2 \int_0^t \langle \phi(x)\phi(y), \mu^{(n)}_{s_n}(dx)\mu^{(n)}_{s_n}(dy)\rangle ds \\
&\quad - \frac{g^2}{\theta^n} \int_0^t \langle \phi^2(x), \mu^{(n)}_{s_n}(dx)\rangle ds \qquad \textit{for every } t \ge 0,
\end{aligned}$$

where $s_n = \lfloor s\theta^n \rfloor/(\theta^n)$ and $\lfloor t \rfloor$ is the maximum integer less or equal to t.

(ii) *For any $t \ge 0$ and $n \ge 1$, we have*

$$\mathbb{E}\left[\sup_{0\le s\le t} \langle 1, \mu^{(n)}_s\rangle^2\right] \le [8t\,\langle 1, \mu^{(n)}_0\rangle + \frac{8}{\theta^n} t g^2 \langle 1, \mu^{(n)}_0\rangle + 2\langle 1, \mu^{(n)}_0\rangle^2] \exp\{8g^2 t\},$$

where $g^2 := \mathbb{E}\xi^2(t)$. Furthermore, there is a constant $c_4 > 0$ such that for every $t \ge 0$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t} \langle 1, \mu^{(n)}_s\rangle^4\right] &\le \left\{16c_4 t^2 (1 + \frac{g^4}{\theta^{2n}}) \left[8t\,\langle 1, \mu^{(n)}_0\rangle + \frac{8}{\theta^n} t g^2 \langle 1, \mu^{(n)}_0\rangle + 2\langle 1, \mu^{(n)}_0\rangle^2\right]\right. \\
&\quad \left. \exp\{8g^2 t\} + 8\langle 1, \mu^{(n)}_0\rangle^4\right\} e^{16c_4 g^4 t}
\end{aligned}$$

(iii) *$\{\mu^{(n)}_t : t \ge 0\}$ defined by (1.6) is tight as a family of processes with sample paths in $D([0,\infty), \mathcal{S}'(\mathbb{R}))$.*

Proof (i) Recall that, for each $n \geq 1$, $\{C_\alpha^{(n)} = \frac{1}{\theta^n} : \alpha \in \Re\}$. For any $k \in \mathbb{N}$ and $s = k/(\theta^n)$ we have

$$\mathbb{P}(O_\alpha^{(n)}(s) = 0|\xi(s)) = \frac{1}{2} - \frac{\xi(s)}{2\sqrt{\theta^n}},$$
$$\mathbb{P}(O_\alpha^{(n)}(s) = 2|\xi(s)) = \frac{1}{2} + \frac{\xi(s)}{2\sqrt{\theta^n}}.$$

Let $\zeta(\alpha) = k/(\theta^n)$ and suppose that $k/(\theta^n) \leq t$. Then, we have

$$\begin{aligned}
&\mathbb{E}\frac{[O_\alpha^{(n)}(\zeta(\alpha)) - 1]}{\theta^n}\phi(x_\alpha(\zeta(\alpha)-))\, 1_{\{\zeta(\alpha)\leq t\}} \\
&= \mathbb{E}\frac{[O_\alpha^{(n)}(\frac{k}{\theta^n}) - 1]}{\theta^n}\phi(x_\alpha(\frac{k}{\theta^n}-)) \\
&= \mathbb{E}\left[\frac{\xi(\frac{k}{\theta^n})}{\theta^{\frac{3n}{2}}}\phi(x_\alpha(\frac{k}{\theta^n}-))\right] \\
&= 0.
\end{aligned}$$

Thus $\mathbb{E}\big\{M_t^{(n)}(\phi)\big\} = 0$ for every $t > 0$ and $\phi \in \mathcal{S}(\mathbb{R})$. Since this is valid for any initial distribution $\mu_0^{(n)}$, by the Markov property of $\{\mu_t^{(n)} : t \geq 0\}$, we have for every $t, s > 0$,

$$\mathbb{E}\left[M_{t+s}^{(n)}(\phi) - M_t^{(n)}(\phi)\big|\mathcal{F}_t^{(n)}\right] = \mathbb{E}_{\mu_t^{(n)}}\left[M_s^{(n)}(\phi) - M_0^{(n)}(\phi)\right] = 0.$$

This shows that $M^{(n)}(\phi)$ is a martingale. Clearly it is purely discontinuous. Now let us find the increasing process $\langle M_t^{(n)}(\phi)\rangle$. First let $[M_t^{(n)}(\phi)]$ be the square variation process of $M_t^{(n)}(\phi)$. Then, we have

$$[M_t^{(n)}(\phi)] = I_1 + I_2, \tag{1.11}$$

where

$$I_1 := \sum_{\alpha\in\Re}\frac{[O_\alpha^{(n)}(\zeta(\alpha)) - 1]^2}{\theta^{2n}}\phi^2(x_\alpha(\zeta(\alpha)-))\, 1_{\{\zeta(\alpha)\leq t\}}$$

and

$$\begin{aligned}
I_2 := \sum_{\alpha,\eta\in\Re;\alpha\neq\eta} \quad &\frac{[O_\alpha^{(n)}(\zeta(\alpha)) - 1][O_\eta^{(n)}(\zeta(\eta)) - 1]}{\theta^{2n}}\phi(x_\alpha(\zeta(\alpha)-)) \\
&\times\phi(x_\eta(\zeta(\eta)-))\, 1_{\{\zeta(\alpha)\leq t\}}1_{\{\zeta(\eta)\leq t\}}.
\end{aligned}$$

Recall $s_n := \lfloor s\theta^n\rfloor/(\theta^n)$. Then, we have

$$\mathbb{E}I_1 = \mathbb{E}\sum_{s_n\leq t}\sum_{\alpha\in\Re}\frac{1}{\theta^{2n}}\phi^2(x_\alpha(s_n-)) = \sum_{s_n\leq t}\mathbb{E}\langle\phi^2, \mu_{s_n}^{(n)}\rangle\frac{1}{\theta^n}$$

and

$$\begin{aligned}
&\mathbb{E}I_2 \\
&= \mathbb{E}\sum_{s_n\leq t}\sum_{\alpha,\eta\in\Re;\alpha\neq\eta}\frac{1}{\theta^{2n}}\mathbb{E}\left\{[O_\alpha^{(n)}(s_n) - 1][O_\eta^{(n)}(s_n) - 1]\Big|\xi(s_n)\right\}\phi(x_\alpha(s_n-))\phi(x_\eta(s_n-)) \\
&= g^2\sum_{s_n\leq t}\mathbb{E}\langle\phi(x)\phi(y), \mu_{s_n}^{(n)}(dx)\mu_{s_n}^{(n)}(dy)\rangle\frac{1}{\theta^n}
\end{aligned}$$

$$-\frac{g^2}{\theta^n}\sum_{s_n\le t}\mathbb{E}\langle\phi^2(x),\mu^{(n)}_{s_n}(dx)\rangle\frac{1}{\theta^n}.$$

Let

$$\begin{aligned}K(\phi)_t \quad &:= \quad \int_0^t\langle\phi^2,\mu^{(n)}_{s_n}\rangle ds\\ &\quad +g^2\int_0^t\langle\phi(x)\phi(y),\mu^{(n)}_{s_n}(dx)\mu^{(n)}_{s_n}(dy)\rangle ds\\ &\quad -\frac{g^2}{\theta^n}\int_0^t\langle\phi^2(x),\mu^{(n)}_{s_n}(dx)\rangle ds \qquad \text{for every } t\ge 0.\end{aligned}$$

First from above calculation we have

$$\mathbb{E}\left[M^{(n)}_t(\phi)^2\right]=\mathbb{E}K(\phi)_t. \tag{1.12}$$

Note that the identity (1.12) holds for any initial distribution $\mu^{(n)}_0$. By the Markov property of $\{\mu^{(n)}_t : t\ge 0\}$ again, we have for every $t, s>0$,

$$\begin{aligned}&\mathbb{E}\Big[M^{(n)}_{t+s}(\phi)^2-M^{(n)}_t(\phi)^2-\{K(\phi)_{t+s}-K(\phi)_t\}\big|\mathcal{F}^{(n)}_t\Big]\\ &=\mathbb{E}_{\mu^{(n)}_t}\Big[M^{(n)}_s(\phi)^2-K(\phi)_s\Big]=0.\end{aligned}$$

This shows that $M^{(n)}_t(\phi)^2-K(\phi)_t$ is a martingale. Hence we conclude that $M^{(n)}(\phi)$ is a purely discontinuous square integrable martingale with

$$\langle M^{(n)}_t(\phi)\rangle=K(\phi)_t.$$

.

(ii) The proof of this part is related to the total number of particles of the system. Since $\langle 1,\mu^{(n)}_t-\mu^{(n)}_0\rangle=M^{(n)}_t(1)$ is a zero-mean martingale, by Doob's maximal inequality and above (i), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le s\le t}\langle 1,\mu^{(n)}_s\rangle^2\Big] \quad &\le \quad 2\mathbb{E}\Big[\sup_{0\le s\le t}M^{(n)}_s(1)^2\Big]+2\langle 1,\mu^{(n)}_0\rangle^2\\ &\le \quad 8\mathbb{E}\left[M^{(n)}_t(1)^2\right]+2\langle 1,\mu^{(n)}_0\rangle^2\\ &\le \quad 8t\,\langle 1,\mu^{(n)}_0\rangle+8g^2\int_0^t\mathbb{E}[\sup_{0\le u\le s}\langle 1,\mu^{(n)}_u\rangle^2]ds\\ &\qquad +\frac{8}{\theta^n}tg^2\langle 1,\mu^{(n)}_0\rangle+2\langle 1,\mu^{(n)}_0\rangle^2.\end{aligned}$$

By Gronwall's inequality, we have

$$\mathbb{E}\Big[\sup_{0\le s\le t}\langle 1,\mu^{(n)}_s\rangle^2\Big] \quad \le \quad [8t\,\langle 1,\mu^{(n)}_0\rangle+\frac{8}{\theta^n}tg^2\langle 1,\mu^{(n)}_0\rangle+2\langle 1,\mu^{(n)}_0\rangle^2]\exp\{8g^2t\}.$$

Note that $M^{(n)}_t(1)$ is a purely discontinuous martingale. Thus, by Burkholder's inequality (See Protter [17] page 222) we have

$$\mathbb{E}\sup_{0\le s\le t}\langle 1,\mu^{(n)}_s\rangle^4 \quad = \quad \mathbb{E}\sup_{0\le s\le t}[\langle 1,\mu^{(n)}_s-\mu^{(n)}_0\rangle+\langle 1,\mu^{(n)}_0\rangle]^4$$

$$
\begin{aligned}
&\leq 8\mathbb{E}\sup_{0\leq s\leq t}\langle 1,\mu_s^{(n)}-\mu_0^{(n)}\rangle^4+8\langle 1,\mu_0^{(n)}\rangle^4\\
&= 8\mathbb{E}\sup_{0\leq s\leq t}\{M_s^{(n)}(1)\}^4+8\langle 1,\mu_0^{(n)}\rangle^4\\
&\leq 8c_4\mathbb{E}\langle M_t^{(n)}(1)\rangle^2+8\langle 1,\mu_0^{(n)}\rangle^4\\
&\leq 16c_4t^2(1+\frac{g^4}{\theta^{2n}})\mathbb{E}\sup_{0\leq s\leq t}\langle 1,\mu_s^{(n)}\rangle^2+8\langle 1,\mu_0^{(n)}\rangle^4\\
&\quad +16c_4g^4\int_0^t\mathbb{E}\sup_{0\leq u\leq s}\langle 1,\mu_u^{(n)}\rangle^4ds.
\end{aligned}
$$

Then, by Gronwall's inequality we have

$$
\mathbb{E}\sup_{0\leq s\leq t}\langle 1,\mu_s^{(n)}\rangle^4 \leq [16c_4t^2(1+\frac{g^4}{\theta^{2n}})\mathbb{E}\sup_{0\leq s\leq t}\langle 1,\mu_s^{(n)}\rangle^2+8\langle 1,\mu_0^{(n)}\rangle^4]e^{16c_4g^4t}.
$$

This gives that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}\langle 1,\mu_s^{(n)}\rangle^4 \leq &\Bigg\{16c_4t^2(1+\frac{g^4}{\theta^{2n}})\Bigg[8t\,\langle 1,\mu_0^{(n)}\rangle+\frac{8}{\theta^n}tg^2\langle 1,\mu_0^{(n)}\rangle+2\langle 1,\mu_0^{(n)}\rangle^2\Bigg]\\
&\exp\{8g^2t\}+8\langle 1,\mu_0^{(n)}\rangle^4\Bigg\}e^{16c_4g^4t}
\end{aligned}
$$

We know that $\mu_0^{(n)}\in M_F(\mathbb{R})$. Therefore, the conclusion follows.

(iii) By Mitoma's Theorem (Mitoma [14]), Theorem 4.5.4 in Dawson [2], and part (ii) above, which implies non-explosion in finite time, we only need to prove that, if we are given $\varepsilon>0$, $T>0$, $\phi\in\mathcal{S}(\mathbb{R})$, and a sequence of stopping times τ_n bounded by T, then $\forall\,\eta>0$, $\exists\,\delta,n_0$ such that $\sup_{n\geq n_0}\sup_{t\in[0,\delta]}\mathbb{P}\{|\mu^{(n)}_{\tau_n+t}(\phi)-\mu^{(n)}_{\tau_n}(\phi)|>\varepsilon\}\leq\eta$.

By (1.10), we have

$$
\begin{aligned}
&\mathbb{P}(|\mu^{(n)}_{\tau_n+t}(\phi)-\mu^{(n)}_{\tau_n}(\phi)|>\varepsilon)\\
&\leq \frac{1}{\varepsilon^2}\mathbb{E}\Bigg[\Big(\mu^{(n)}_{\tau_n+t}(\phi)-\mu^{(n)}_{\tau_n}(\phi)\Big)^2\Bigg]\\
&= \frac{1}{\varepsilon^2}\mathbb{E}\Bigg[\Big(M^{(n)}_{\tau_n+t}(\phi)-M^{(n)}_{\tau_n}(\phi)\Big)^2\Bigg].
\end{aligned}
$$

Note that we have by part (i) of this lemma that

$$
\begin{aligned}
\mathbb{E}\Bigg[\Big(M^{(n)}_{\tau_n+t}(\phi)-M^{(n)}_{\tau_n}(\phi)\Big)^2\Bigg] \leq &\|\phi^2\|_\infty\,t(1+\frac{g^2}{\theta^n})\,\mathbb{E}\Bigg[\sup_{s\leq T+t}\langle 1,\mu_s^{(n)}\rangle\Bigg]\\
&+\|\phi\|_\infty^2\,tg^2\,\mathbb{E}\Bigg[\sup_{s\leq T+t}\langle 1,\mu_s^{(n)}\rangle^2\Bigg].
\end{aligned}
$$

Therefore by part (ii) of this lemma and Lemma 3.4 of Wang [20], we conclude that for every $\varepsilon>0$, there is a constant $c>0$ such that

$$
\sup_{n\geq 1}\sup_{t\in[0,\delta]}\mathbb{P}\left(|\mu^{(n)}_{\tau_n+t}(\phi)-\mu^{(n)}_{\tau_n}(\phi)|>\varepsilon\right)\leq c\delta \qquad \text{for every } \delta>0,
$$

which proves (iii). This completes the proof of the lemma. □

Theorem 1.3 *Given any $\mu\in M_F(\mathbb{R})$, set $\{\mu_0^{(n)}=\mu,n\in\mathbb{N}\}$. Then, any weak limit, denoted by $\{\mu_t,t\geq 0\}$, of a convergent subsequence of above constructed $\{\mu_t^{(n)},t\geq 0,n\in\mathbb{N}\}$ is a solution to $(\mathcal{B},\delta_\mu)-$martingale problem.*

2 Dual Stochastic Evolution Equations and Approximation

Let $\{\mu_t, t \geq 0\}$ be a solution to $(\mathcal{B}, \delta_\mu)-$martingale problem constructed in the previous section. In the following, we will construct a conditional log-Laplace functional duality (CLLF) for the branching process $\{x_t :=< 1, \mu_t >, t \geq 0\}$. As a byproduct, this also proves the uniqueness of $(\mathcal{B}, \delta_\mu)-$martingale problem. Similar to the construction of the conditional log-Laplace functional duality given by Li et al. [13] and Xiong [21], we consider following backward stochastic evolution equation for the dual process of $\{x_t, t \geq 0\}$:

$$u_{s,t} = \lambda - \frac{1}{2}\int_s^t u_{r,t}^2 dr + g\int_s^t u_{r,t} \cdot dW_r, \tag{2.1}$$

where λ and g are non-negative constants and the last term is a backward Itô's integral with respect to the standard one-dimensional Brownian motion W. In the right hand side of the equation (2.1), the first term is the initial value; the second term describes the dual branching dynamic and the last term represents the dual branching interaction dynamic generated by the random environments. For our convenience, we consider the corresponding forward equation:

$$u_t = \lambda - \frac{1}{2}\int_0^t u_r^2 dr + g\int_0^t u_r dW_r, \tag{2.2}$$

where $\lambda \geq 0$ is a constant.

Theorem 2.1 *For any given nonnegative constants $\lambda \geq 0$ and $0 \leq g \leq 1$, the equation (2.2) has a unique nonnegative solution which is a diffusion process.*

Proof Since the initial value λ is nonnegative, the solution of equation (2.2) is nonnegative (See [9] Example 8.2 pp.221). The remaining conclusions follow from Theorem 3.1 of Ikeda-Watanabe ([9] pp.164). $\square$

In order to construct the conditional log-Laplace functional duality, we consider the Wong-Zakai type approximation to the forward stochastic evolution equation (2.2):

$$u_t^\epsilon = \lambda - \frac{1}{2}\int_0^t (u_r^\epsilon)^2 dr - \frac{g^2}{2}\int_0^t u_r^\epsilon dr + g\int_0^t u_r^\epsilon \dot{W}_r^\epsilon dr, \tag{2.3}$$

where $\lambda \geq 0$, $0 \leq g \leq 1$ are nonnegative constants and for any $\epsilon > 0$ and any $i \in \mathbb{N}$,

$$\dot{W}_r^\epsilon := \frac{1}{\epsilon}\{W_{(i+1)\epsilon} - W_{i\epsilon}\}, \qquad \text{for } i\epsilon \leq r < (i+1)\epsilon.$$

In the right hand side of (2.3), the third term comes from the correction of the Wong-Zakai approximation to the equation (2.2).

Theorem 2.2 *For any given nonnegative constants $\lambda \geq 0$ and $0 \leq g \leq 1$, the equation (2.3) has a unique nonnegative solution.*

Proof The corresponding differential equation of (2.3) is

$$du_r^\epsilon = -\frac{1}{2}(u_r^\epsilon)^2 dr - \frac{g^2}{2}u_r^\epsilon dr + g u_r^\epsilon \dot{W}_r^\epsilon dr, \qquad \text{with } u_0^\epsilon = \lambda. \tag{2.4}$$

Let $v_t = (u_t^\epsilon)^{-1}$. Then (2.4) is changed into following form:

$$\frac{dv_t}{dt} - (\frac{1}{2}g^2 - g\dot{W}_t^\epsilon)v_t = \frac{1}{2}. \tag{2.5}$$

Define

$$h_t := \exp\{-\int_0^t (\frac{1}{2}g^2 - g\dot{W}_r^\epsilon)dr\}$$

and multiply h_t on the both sides of (2.5). Then, we get

$$h_t\frac{dv_t}{dt} - (\frac{1}{2}g^2 - g\dot{W}_t^\epsilon)h_t v_t = \frac{1}{2}h_t \tag{2.6}$$

or

$$d(h_t v_t) = \frac{1}{2}h_t dt$$

which gives

$$\begin{aligned} u_t^\epsilon &= h_t\{\frac{1}{\lambda} + \frac{1}{2}\int_0^t h_s ds\}^{-1} \\ &= \frac{\exp\{-\int_0^t(\frac{1}{2}g^2 - g\dot{W}_r^\epsilon)dr\}}{\frac{1}{\lambda} + \frac{1}{2}\int_0^t \exp\{-\int_0^s(\frac{1}{2}g^2 - g\dot{W}_r^\epsilon)dr\}ds}. \end{aligned} \tag{2.7}$$

Since the equation (2.4) satisfies the conditions of the uniqueness theorem for the solution of ordinary differential equation, the equation (2.4) with initial value λ has a unique solution. □

Theorem 2.3 *For any given nonnegative constants $\lambda \geq 0$ and $0 \leq g \leq 1$, let $\{u_t, t \geq 0\}$ be the solution of the equation (2.2) and $\{u_t^\epsilon, t \geq 0\}$ be the solution of equation (2.3). Then, for any $t \geq 0$,*

$$\lim_{\epsilon\to 0}\mathbb{E}(u_t - u_t^\epsilon)^2 = 0.$$

Proof Let $z_t = u_t - u_t^\epsilon$. Then, according to equation (2.2) and equation (2.3), $\{z_t, t \geq 0\}$ is a solution of following equation:

$$\begin{aligned} z_t &= -\frac{1}{2}\int_0^t (u_r + u_r^\epsilon)z_r dr + g\int_0^t u_r dW_r + \frac{g^2}{2}\int_0^t u_r^\epsilon dr - g\int_0^t u_r^\epsilon \dot{W}_r^\epsilon dr \\ &= \int_0^t \{-\frac{1}{2}(u_r + u_r^\epsilon)z_r + \frac{g^2}{2}u_r^\epsilon - g u_r^\epsilon \dot{W}_r^\epsilon\}dr + g\int_0^t u_r dW_r. \end{aligned} \tag{2.8}$$

By Itô's formula, we have

$$z_t^2 = \int_0^t 2z_r g u_r dW_r + \int_0^t 2z_r\{-\frac{1}{2}(u_r + u_r^\epsilon)z_r + \frac{g^2}{2}u_r^\epsilon - g u_r^\epsilon \dot{W}_r^\epsilon\}dr + \int_0^t g^2 u_r^2 dr. \tag{2.9}$$

For $i\epsilon \leq r$, let $f(x,y) = xy$, by Itô's formula again (Since Itô's formula requires adaptation, we need to replace u_s^ϵ and $\dot{W}_s^\epsilon$ by $u_{s-\epsilon}^\epsilon$ and $\dot{W}_{s-\epsilon}^\epsilon$, respectively. However, to simplify notation, we still keep u_s^ϵ.), we have

$$\begin{aligned} z_r u_r^\epsilon - z_{(i-1)\epsilon}u_{(i-1)\epsilon}^\epsilon &= \int_{(i-1)\epsilon}^r u_s^\epsilon g u_s dW_s + \int_{(i-1)\epsilon}^r z_s\{-\frac{1}{2}(u_s^\epsilon)^2 - \frac{g^2}{2}u_s^\epsilon + g u_s^\epsilon \dot{W}_{s-\epsilon}^\epsilon\}ds \\ &+ \int_{(i-1)\epsilon}^r u_s^\epsilon\{-\frac{1}{2}(u_s + u_s^\epsilon)z_s + \frac{g^2}{2}u_s^\epsilon - g u_s^\epsilon \dot{W}_{s-\epsilon}^\epsilon\}ds. \end{aligned} \tag{2.10}$$

In the following, in order to use Gronwall's inequality, we will estimate each term of the right hand side of equation (2.9). We may assume that $t = k\epsilon$. Then, we have

$$\mathbb{E}\int_0^t 2z_r(-g u_r^\epsilon)\dot{W}_{r-\epsilon}^\epsilon dr$$

$$
\begin{aligned}
&= (-2g)\mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon} z_r u_r^\epsilon \dot{W}_{r-\epsilon}^\epsilon dr \\
&= (-2g)\mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon} (z_r u_r^\epsilon - z_{(i-1)\epsilon}u_{(i-1)\epsilon}^\epsilon)\dot{W}_{r-\epsilon}^\epsilon dr. \qquad (2.11)
\end{aligned}
$$

Now we use (2.10) to estimate (2.11). In the following, if we write $x \approx y$ we mean $|x-y| \le K\sqrt{\epsilon}$. Let us consider one term.

$$
\begin{aligned}
(-2g) \quad & \mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon}\int_{(i-1)\epsilon}^{r}\{-g(u_s^\epsilon)^2\dot{W}_{s-\epsilon}^\epsilon + z_s g u_s^\epsilon \dot{W}_{s-\epsilon}^\epsilon\}ds\dot{W}_{r-\epsilon}^\epsilon dr \\
\approx \quad & (-2g^2)\mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon}\int_{(i-1)\epsilon}^{i\epsilon}\{[-(u_{(i-2)\epsilon}^\epsilon)^2 + z_{(i-2)\epsilon}u_{(i-2)\epsilon}^\epsilon]\dot{W}_{s-\epsilon}^\epsilon\}ds\dot{W}_{r-\epsilon}^\epsilon dr \\
& -2g^2\mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon}\int_{i\epsilon}^{r}\{-(u_{(i-2)\epsilon}^\epsilon)^2 + z_{(i-2)\epsilon}u_{(i-2)\epsilon}^\epsilon\}dsdr\epsilon^{-2}[W_{i\epsilon} - W_{(i-1)\epsilon}]^2 \\
= \quad & (-g^2)\sum_{i=0}^{k-1}\epsilon^2\mathbb{E}\{-(u_{(i-2)\epsilon}^\epsilon)^2 + z_{(i-2)\epsilon}u_{(i-2)\epsilon}^\epsilon\}\epsilon^{-2}\epsilon \\
\approx \quad & -g^2\mathbb{E}\int_0^t\{-(u_u^\epsilon)^2 + z_u u_u^\epsilon\}du. \qquad (2.12)
\end{aligned}
$$

Now consider another term,

$$
\begin{aligned}
(-2g) \quad & \mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon}\int_{(i-1)\epsilon}^{r}\{g u_s^\epsilon u_s\}dW_s\dot{W}_{r-\epsilon}^\epsilon dr \\
\approx \quad & (-2g^2)\mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon}\int_{(i-1)\epsilon}^{i\epsilon}u_{(i-2)\epsilon}^\epsilon u_{(i-2)\epsilon}dW_s\epsilon^{-1}[W_{i\epsilon} - W_{(i-1)\epsilon}]dr \\
& -2g^2\mathbb{E}\sum_{i=0}^{k-1}\int_{i\epsilon}^{(i+1)\epsilon}\int_{i\epsilon}^{r}\{u_{(i-2)\epsilon}^\epsilon u_{(i-2)\epsilon}\}dW_s\epsilon^{-1}[W_{i\epsilon} - W_{(i-1)\epsilon}]dr \\
= \quad & (-2g^2)\sum_{i=0}^{k-1}\mathbb{E}\{u_{(i-2)\epsilon}u_{(i-2)\epsilon}^\epsilon\}\epsilon \\
\approx \quad & -2g^2\mathbb{E}\int_0^t\{u_u^\epsilon u_u\}du. \qquad (2.13)
\end{aligned}
$$

Now combine (2.12), (2.13), and (2.9), we have

$$
\begin{aligned}
\mathbb{E}z_t^2 \quad \le \quad & \mathbb{E}\int_0^t\{-z_r^2(u_r + u_r^\epsilon) + g^2 z_r u_r^\epsilon + g^2 u_r^2 - 2g^2 u_r u_r^\epsilon - g^2 u_r u_r^\epsilon \\
& +2g^2(u_r^\epsilon)^2\}dr + K\sqrt{\epsilon} \\
= \quad & \mathbb{E}\int_0^t\{-g^2 u_r u_r^\epsilon + 2g^2(u_r^\epsilon)^2 - 2g^2 u_r u_r^\epsilon - z_r^2(u_r + u_r^\epsilon) \\
& +g^2 u_r u_r^\epsilon - g^2(u_r^\epsilon)^2 + g^2 u_r^2\}dr + K\sqrt{\epsilon} \\
= \quad & \mathbb{E}\int_0^t\{g^2 u_r(u_r - u_r^\epsilon) + g^2 u_r^\epsilon(u_r^\epsilon - u_r) - z_r^2(u_r + u_r^\epsilon)\}dr + K\sqrt{\epsilon} \\
\le \quad & g^2\int_0^t\mathbb{E}z_r^2 dr + K\sqrt{\epsilon}. \qquad (2.14)
\end{aligned}
$$

Then, the conclusion follows from the Gronwall's inequality. □

Theorem 2.4 *For any given nonnegative constants $\lambda \geq 0$ and $0 \leq g \leq 1$, let $\{u_t, t \geq 0\}$ be the solution of the equation (2.2) and $\{u_t^\epsilon, t \geq 0\}$ be the solution of equation (2.3). Then, for any $t \geq 0$, we have $\mathbb{E}u_t \leq \lambda$,*

$$0 \leq u_t \leq \lambda \exp\{gW_t - \frac{1}{2}g^2 t\}$$

and

$$0 \leq u_t^\epsilon \leq \lambda \exp\{g\int_0^t \dot{W}_r^\epsilon dr - \frac{1}{2}g^2 t\}.$$

For $k\epsilon \leq t < (k+1)\epsilon$, we have

$$gW_{k\epsilon} - g|W_{(k+1)\epsilon} - W_{k\epsilon}| \leq g\int_0^t \dot{W}_r^\epsilon dr \leq gW_{k\epsilon} + g|W_{(k+1)\epsilon} - W_{k\epsilon}|.$$

Proof According to Theorem 2.3, we may assume that u_t^ϵ converges to u_t almost surely. From (2.7), we have

$$u_t = \frac{\exp\{gW_t - \frac{1}{2}g^2 t\}}{\frac{1}{\lambda} + \frac{1}{2}\int_0^t \exp\{gW_s - \frac{1}{2}g^2 s\}ds}. \tag{2.15}$$

Since $\exp\{gW_t - \frac{1}{2}g^2 t\}$ is an exponential martingale, we have $\mathbb{E}u_t \leq \lambda$. Remaining conclusions are obvious. □

3 Conditional Martingale Problems and Convergence

Let $\mathbb{P}^W$ be the conditional probability measure given W and let x_t^ϵ be a solution to the following conditional martingale problem: x_t^ϵ is a real-valued process such that for any given $x_0 \geq 0$

$$M_t^\epsilon \equiv x_t^\epsilon - x_0 - \int_0^t (gx_r^\epsilon \dot{W}_r^\epsilon - \frac{g^2}{2}x_r^\epsilon))dr \tag{3.1}$$

is a continuous $\mathbb{P}^W$-martingale with quadratic variation process

$$\langle M^\epsilon \rangle_t = \int_0^t x_r^\epsilon dr.$$

By Theorem 2.6 of Ethier-Kurtz ([7] pp.374), for each given $\epsilon > 0$ above conditional martingale problem has unique solution $\{x_t^\epsilon, t \geq 0\}$.

Theorem 3.1 *For any given nonnegative constants $\lambda \geq 0$ and $0 \leq g \leq 1$, there exists a constant K_1 which is independent of ϵ such that*
(i)

$$\mathbb{E}\sup_{0\leq s\leq t}(x_s^\epsilon)^4 \leq K_1$$

and there exists a constant K_2 which is independent of t, s and ϵ such that
(ii)

$$\mathbb{E}(x_t^\epsilon - x_s^\epsilon)^4 \leq K_2|t-s|^2.$$

(iii) In particular, (i) and (ii) imply the tightness of $\{x_t^\epsilon, t \geq 0\}$ in $C([0,\infty), \mathbb{R})$.

Proof First we consider (i). Let x_t^ϵ be a solution to the following conditional martingale problem: x_t^ϵ is a real-valued process such that for any given $x_0 \geq 0$

$$M_t^\epsilon \equiv x_t^\epsilon - x_0 - \int_0^t (gx_r^\epsilon \dot{W}_r^\epsilon - \frac{g^2}{2} x_r^\epsilon))dr \tag{3.2}$$

is a continuous $\mathbb{P}^W$-martingale with quadratic variation process

$$\langle M^\epsilon \rangle_t = \int_0^t x_r^\epsilon dr.$$

According to the martingale representation theorem, there exists a Brownian motion $\{B_t, t \geq 0\}$ which is independent of the Brownian motion $\{W_t, t \geq 0\}$ such that for any constant $x_0 \geq 0$, the following equation

$$x_t^\epsilon = x_0 + \int_0^t (gx_r^\epsilon \dot{W}_r^\epsilon - \frac{g^2}{2} x_r^\epsilon)dr + \int_0^t \sqrt{x_r^\epsilon} dB_r \tag{3.3}$$

holds. This equation has a non-negative solution ((See [9] Example 8.2 pp.221)). Then, according to Itô's formula, we have following equation:

$$(x_t^\epsilon)^4 = (x_0)^4 + \int_0^t [4(x_r^\epsilon)^4(g\dot{W}_r^\epsilon - \frac{g^2}{2}) + 6(x_r^\epsilon)^3]dr + \int_0^t 4(x_t^\epsilon)^3 \sqrt{x_r^\epsilon} dB_r. \tag{3.4}$$

Let us give following notation:

$$A := \int_0^t [(x_r^\epsilon)^4 \dot{W}_r^\epsilon]dr,$$

$$B := \int_0^t (x_t^\epsilon)^3 \sqrt{x_r^\epsilon} dB_r$$

and

$$C := \int_0^t [(x_r^\epsilon)^3]dr.$$

Then, according to (3.4), we have following inequality:

$$(x_t^\epsilon)^4 \leq (x_0)^4 + 4g|A| + 4B + 6C. \tag{3.5}$$

Let $t = (k+1)\epsilon$. First, it is easy to see the following estimation:

$$\mathbb{E}C \leq 6\mathbb{E} \int_0^t [(x_r^\epsilon)^4]dr + 4t.$$

Since B is a martingale, by moment inequality of martingale, for any $\delta > 0$, we have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\int_0^s (x_r^\epsilon)^3 \sqrt{x_r^\epsilon} dB_r| &\leq \mathbb{E}\sqrt{\int_0^t (x_r^\epsilon)^7 dr} \\ &\leq \mathbb{E}\sqrt{\delta \sup_{0 \leq u \leq t} (x_u^\epsilon)^4} \sqrt{\delta^{-1} \int_0^t (x_r^\epsilon)^3 dr} \\ &\leq 2\delta \mathbb{E} \sup_{0 \leq u \leq t} (x_u^\epsilon)^4 + 6\delta^{-1} \int_0^t \mathbb{E}(x_r^\epsilon)^4 dr + 4t. \end{aligned}$$

Then, we have

$$A = Aa(k) + Ab(k),$$

where

$$Aa(k) := \sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} \{(x_r^\epsilon)^4 - (x_{i\epsilon}^\epsilon)^4\}\epsilon^{-1}[W_{(i+1)\epsilon} - W_{i\epsilon}]dr$$

and

$$Ab(k) := \sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} (x_{i\epsilon}^\epsilon)^4 \epsilon^{-1}[W_{(i+1)\epsilon} - W_{i\epsilon}]dr.$$

Thus, by moment inequality for martingale there exists a positive constant K such that

$$\begin{aligned}
\mathbb{E} \sup_{0\le j\le k} |Ab(j)| &\le \mathbb{E} \sup_{0\le j\le k} |\sum_{i=0}^{j} (x_{i\epsilon}^\epsilon)^4 [W_{(i+1)\epsilon} - W_{i\epsilon}]| \\
&\le \mathbb{E}\sqrt{\int_0^t (x_r^\epsilon)^8 dr} + K \\
&\le \mathbb{E}\sqrt{\delta \sup_{0\le u\le t} (x_u^\epsilon)^4}\sqrt{\delta^{-1}\int_0^t (x_r^\epsilon)^4 dr} + K \\
&\le 4\delta \mathbb{E} \sup_{0\le u\le t} (x_u^\epsilon)^4 + 4\delta^{-1}\mathbb{E}\int_0^t (x_r^\epsilon)^4 dr + K. \qquad (3.6)
\end{aligned}$$

For the $Aa(k)$ part, using (3.4) we get

$$Aa(k) = (4g)Aaa(k) + 4Aab(k) + 6Aac(k) - \frac{g^2}{2}Aad(k),$$

where

$$Aaa(k) := \sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} \int_{i\epsilon}^{r} (x_u^\epsilon)^4 du \epsilon^{-2}[W_{(i+1)\epsilon} - W_{i\epsilon}]^2 dr,$$

$$Aab(k) := \sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} \int_{i\epsilon}^{r} (x_u^\epsilon)^{7/2} dB_u \epsilon^{-1}[W_{(i+1)\epsilon} - W_{i\epsilon}]dr,$$

$$Aac(k) := \sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} \int_{i\epsilon}^{r} (x_u^\epsilon)^3 du \epsilon^{-1}[W_{(i+1)\epsilon} - W_{i\epsilon}]dr$$

and

$$Aad(k) := \sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} \int_{i\epsilon}^{r} (x_u^\epsilon)^4 du \epsilon^{-1}[W_{(i+1)\epsilon} - W_{i\epsilon}]dr.$$

Then, we have

$$\begin{aligned}
\mathbb{E} \sup_{0\le j\le k} Aaa(j) &\le \mathbb{E}\sum_{i=0}^{k} \int_{i\epsilon}^{(i+1)\epsilon} \int_{i\epsilon}^{r} (x_u^\epsilon)^4 du \epsilon^{-2}[W_{(i+1)\epsilon} - W_{i\epsilon}]^2 dr \\
&\approx \mathbb{E}\sum_{i=0}^{k} (x_{i\epsilon}^\epsilon)^4 [W_{(i+1)\epsilon} - W_{i\epsilon}]^2 \\
&\approx \mathbb{E}\int_0^t (x_u^\epsilon)^4 du.
\end{aligned}$$

Since

$$\begin{aligned} |Aab(k)| &= |\sum_{i=0}^{k}\int_{i\epsilon}^{(i+1)\epsilon}\int_{i\epsilon}^{r}(x_u^\epsilon)^{7/2}dB_u\epsilon^{-1}[W_{(i+1)\epsilon}-W_{i\epsilon}]dr| \\ &= \epsilon^{-1}|\sum_{i=0}^{k}\int_{i\epsilon}^{(i+1)\epsilon}[(i+1)\epsilon-u](x_u^\epsilon)^{7/2}dB_u[W_{(i+1)\epsilon}-W_{i\epsilon}]| \\ &\le \epsilon^{-1}\sum_{i=0}^{k}\sup_{i\epsilon\le s\le(i+1)\epsilon}|\int_{i\epsilon}^{s}[(i+1)\epsilon-u](x_u^\epsilon)^{7/2}dB_u\int_{i\epsilon}^{s}dW_u|, \end{aligned}$$

by the moment inequality and the inequality (2.12) of Ikeda-Watanabe ([9] pp.56), for any $\delta>0$, we have

$$\begin{aligned} \mathbb{E}\sup_{0\le j\le k}|Aab(j)| &\le \epsilon^{-1}\mathbb{E}\sum_{i=0}^{k}\sqrt{\epsilon}\sqrt{\int_{i\epsilon}^{(i+1)\epsilon}[(i+1)\epsilon-u]^2(x_u^\epsilon)^7du} \\ &\le \sum_{i=0}^{k}\mathbb{E}\sqrt{\epsilon\delta\sup_{i\epsilon\le r\le(i+1)\epsilon}(x_r^\epsilon)^4}\sqrt{\delta^{-1}\int_{i\epsilon}^{(i+1)\epsilon}(x_u^\epsilon)^3du} \\ &\le 2t\delta\mathbb{E}\sup_{0\le r\le t}(x_r^\epsilon)^4+2\delta^{-1}\mathbb{E}\int_0^t(x_u^\epsilon)^3du \\ &\le 2t\delta\mathbb{E}\sup_{0\le r\le t}(x_r^\epsilon)^4+12\delta^{-1}\mathbb{E}\int_0^t(x_u^\epsilon)^4du+4t. \end{aligned}$$

For remaining two terms, the idea is same. We only estimate one term.

$$\begin{aligned} \mathbb{E}\sup_{0\le j\le k}|Aac(j)| &\le \mathbb{E}\sum_{i=0}^{k}\int_{i\epsilon}^{(i+1)\epsilon}\int_{i\epsilon}^{r}(x_u^\epsilon)^3du\epsilon^{-1}|W_{(i+1)\epsilon}-W_{i\epsilon}|dr \\ &\approx \mathbb{E}\sum_{i=0}^{k}\int_{i\epsilon}^{(i+1)\epsilon}\int_{i\epsilon}^{r}(x_{i\epsilon}^\epsilon)^3du\epsilon^{-1}|W_{(i+1)\epsilon}-W_{i\epsilon}|dr \\ &\le \mathbb{E}\sum_{i=0}^{k}\int_{i\epsilon}^{(i+1)\epsilon}\int_{i\epsilon}^{r}(x_{i\epsilon}^\epsilon)^3du\{2\epsilon^{-1}+2\frac{|W_{(i+1)\epsilon}-W_{i\epsilon}|^2}{\epsilon}\}dr \\ &= (1+\epsilon)\int_0^t\mathbb{E}(x_u^\epsilon)^3du \\ &\le 6(1+\epsilon)\int_0^t\mathbb{E}(x_u^\epsilon)^4du+4t. \end{aligned}$$

Then, combining above estimations and choosing δ small enough, we get (i) by Gronwall's inequality.

Now let us consider (ii). From the definition of $\dot{W}_r^\epsilon$ and the adaptation requirement in Itô's integral, we should modify (3.3) as follows:

$$x_t^\epsilon = x_0+\int_0^t(gx_{r-\epsilon}^\epsilon\dot{W}_r^\epsilon-\frac{g^2}{2}x_{r-\epsilon}^\epsilon)dr+\int_0^t\sqrt{x_{r-\epsilon}^\epsilon}dB_r. \tag{3.7}$$

Then, according to (3.7), for any $0\le s<t$, we have following inequality:

$$\mathbb{E}(x_t^\epsilon-x_s^\epsilon)^4\le 64g^4\mathbb{E}(I_1)^4+4g^8\mathbb{E}(I_2)^4+8\mathbb{E}(I_3)^4,$$

where

$$I_1 = \int_s^t x^\epsilon_{r-\epsilon} \dot{W}^\epsilon_r dr,$$

$$I_2 = \int_s^t x^\epsilon_{r-\epsilon} dr$$

and

$$I_3 = \int_s^t \sqrt{x^\epsilon_{r-\epsilon}} dB_r.$$

First,we have that

$$\begin{aligned}
\mathbb{E}(I_3)^4 &\leq c_1 \mathbb{E}[\int_s^t x^\epsilon_{r-\epsilon} dr]^2 \\
&\leq c_1 (t-s)\mathbb{E}\int_s^t (x^\epsilon_{r-\epsilon})^2 dr \\
&\leq c_1 |t-s|^2 \mathbb{E} \sup_{0\leq u\leq t} (x^\epsilon_u)^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(I_2)^4 &\leq \mathbb{E}[(t-s)\int_s^t (x^\epsilon_{r-\epsilon})^2 dr]^2 \\
&\leq |t-s|^4 \mathbb{E} \sup_{0\leq u\leq t} (x^\epsilon_u)^4.
\end{aligned}$$

In the following, since $\epsilon > 0$ is small, we may assume there are positive integers k and l such that $s = k\epsilon$ and $t = (k+l)\epsilon$ (for the case that $t - s \leq \epsilon$, the idea is same for the proof.)

$$\begin{aligned}
\mathbb{E}(I_1)^4 &= \mathbb{E}\Big[\sum_{i=k}^{k+l} \int_{i\epsilon}^{(i+1)\epsilon} x^\epsilon_{r-\epsilon} \dot{W}^\epsilon_r dr\Big]^4 \\
&= II_1 + II_2,
\end{aligned}$$

where

$$\begin{aligned}
II_1 &:= \mathbb{E}\sum_{i=k}^{k+l} \Big(\int_{(i-1)\epsilon}^{i\epsilon} x^\epsilon_r dr\Big)^4 \Big[\epsilon^{-1}(W_{(i+1)\epsilon} - W_{i\epsilon})\Big]^4 \\
&\leq 3\mathbb{E}\sum_{i=k}^{k+l} \sup_{0\leq r\leq t} (x^\epsilon_r)^4 \epsilon^2 \\
&\leq 3(t-s)^2 \mathbb{E} \sup_{0\leq r\leq t} (x^\epsilon_r)^4
\end{aligned}$$

and

$$\begin{aligned}
II_2 &:= \mathbb{E}\sum_{i,j=k,i\neq j}^{k+l} \Big(\int_{(i-1)\epsilon}^{i\epsilon} x^\epsilon_r dr\Big)^2 \Big[\epsilon^{-1}(W_{(i+1)\epsilon} - W_{i\epsilon})\Big]^2 \Big(\int_{(j-1)\epsilon}^{j\epsilon} x^\epsilon_r dr\Big)^2 \Big[\epsilon^{-1}(W_{(j+1)\epsilon} - W_{j\epsilon})\Big]^2 \\
&\leq (t-s)\mathbb{E}\sum_{i=k}^{k+l} \sup_{0\leq r\leq t} (x^\epsilon_r)^4 \epsilon \\
&\leq (t-s)^2 \mathbb{E} \sup_{0\leq r\leq t} (x^\epsilon_r)^4.
\end{aligned}$$

We are done. $\square$

4 Conditional log-Laplace Functional and Uniqueness

Theorem 4.1 *With above notation, we have*

$$\mathbb{E}^W e^{-\lambda x_t^\epsilon} = e^{-x_0 u_t^\epsilon}, \qquad \textit{for } t \geq 0, \tag{4.1}$$

where $x_0^\epsilon = x_0 \geq 0$ *and* $u_0^\epsilon = \lambda \geq 0$.

Proof From duality argument of Ethier-Kurtz ([7] pp.190), we choose $\alpha \equiv 0$, $\beta \equiv 0$, $g(x, y) = h(x, y)$ and

$$\begin{aligned} h(x, y) &= e^{-xy}[xgy\dot{W}_r^\epsilon - \frac{g^2}{2}xy - \frac{1}{2}xy^2], & (4.2)\\ f(x, y) &= e^{-xy}, & (4.3)\end{aligned}$$

where $x \geq 0$ and $y \geq 0$. Then, by (2.3), (3.2) and Itô's formula,

$$M_t^u \equiv f(x_0, u_t^\epsilon) - f(x_0, \lambda) - \int_0^t h(x_0, u_r^\epsilon)dr$$

is a trivial martingale with $\langle M^u \rangle_t = 0$ and

$$M_t^x \equiv f(x_t^\epsilon, \lambda) - f(x_0, \lambda) - \int_0^t h(x_r^\epsilon, \lambda)dr$$

is a $\mathcal{F}_t^{x^\epsilon}$-martingale. It is easy to check that the conditions of Theorem 4.11 of Ethier-Kurtz ([7] p192) are satisfied. Thus, (4.36) of Ethier-Kurtz ([7] pp.189) gives (4.1). $\square$

Theorem 4.2 *With above notation, we have*

$$\mathbb{E}^W e^{-\lambda x_t} = e^{-x_0 u_t}, \qquad \textit{for } t \geq 0, \tag{4.4}$$

where $x_0^\epsilon = x_0 \geq 0$ *and* $u_0^\epsilon = \lambda \geq 0$.

Proof First, we have to prove the conditional uniform integrability of $\{e^{-x_t^{\epsilon_n}}, n \geq 1\}$ with respect to $\mathbb{P}^W$, where $\{\epsilon_n\}$ is any decreasing sequence with $\lim_{n\to\infty} \epsilon_n = 0$. In fact, given W, $u_t^{\epsilon_n}$ is a non-stochastic or deterministic function. Keep this in mind, then we have

$$\begin{aligned} \int_{(e^{-x_t^{\epsilon_n}} \geq k)} e^{-x_t^{\epsilon_n}} d\mathbb{P}^W &\leq \sqrt{\int_\Omega e^{-2x_t^{\epsilon_n}} d\mathbb{P}^W} \sqrt{\mathbb{P}^W(e^{-x_t^{\epsilon_n}} \geq k)} \\ &\leq \sqrt{e^{-x_0 u_t^{\epsilon_n}}} \sqrt{\frac{1}{k^2} \int_\Omega e^{-2x_t^{\epsilon_n}} d\mathbb{P}^W} \\ &\leq \sqrt{e^{-x_0 u_t^{\epsilon_n}}} \sqrt{\frac{1}{k^2} e^{-x_0 u_t^{\epsilon_n}}} \\ &\leq \frac{1}{k} e^{-x_0 u_t^{\epsilon_n}} \\ &\leq \frac{1}{k}, \end{aligned} \tag{4.5}$$

where $u_0^{\epsilon_n} = 2$. This proves the conditional uniform integrability of $\{e^{-x_t^{\epsilon_n}}, n \geq 1\}$ with respect to $\mathbb{P}^W$. Based on Skorohod representation theorem, there exists a probability space such that on it, W and B are still two independent standard Brownian motions and $x_t^{\epsilon_n} \to x_t$ almost surely. Thus, we have

$$\begin{aligned} \mathbb{E}^W e^{-\lambda x_t} &= \lim_{n\to\infty} \mathbb{E}^W e^{-\lambda x_t^{\epsilon_n}} \\ &= \lim_{n\to\infty} e^{-x_0 u_t^{\epsilon_n}} = e^{-x_0 u_t}, \qquad \text{for } t \geq 0, \end{aligned} \tag{4.6}$$

□

In the following, we will use the conditional log-Laplace functional to prove that the martingale problem for $\mathcal{B}$ discussed in Section 1 is well posed.

Theorem 4.3 *Given any $\mu \in M_F(\mathbb{R})$, let $\{\mu_t, t \geq 0\}$ be a solution to $(\mathcal{B}, \delta_\mu)-$martingale problem. Then, $(\mathcal{B}, \delta_\mu)-$martingale problem is well posed.*

Proof From Theorem 4.2, we have

$$\mathbb{E}e^{-\lambda x_t} = \mathbb{E}e^{-x_0 u_t}$$

which combining the Proposition 4.7 (Ethier-Kurtz [7] pp.189) implies the uniqueness of the $(\mathcal{B}, \delta_\mu)-$martingale problem. This completes the proof of Theorem 4.3. □

Acknowledgements The author would like to thank the referee's time and comments.

References

[1] Böinghoff, C. and Hutzenthaler, M.: Branching diffusions in random environment. *Markov Processes Relat. Fields* **18**, 269–310 (2012)

[2] Dawson, D. A.: Infinitely divisible random measures and superprocesses. in *Proc. 1990 Workshop on Stochastic Analysis and Related Topics,* Silivri, Turkey, 1992

[3] Dawson, D. A.: Measure-valued Markov processes. *Lecture Notes in Math.* **1541** 1–260. Springer, Berlin, 1993

[4] Dawson, D. A. and Kurtz, T. G.: Application of duality to measure-valued diffusion processes. *Lecture Notes in Control and Inf. Sci.* **42**, 177–191 (1982)

[5] Dawson, D. A., Li, Z. and Wang, H.: Superprocesses with dependent spatial motion and general branching densities. *Electron. J. Probab.* **6(25)**, 1–33 (2001)

[6] Dynkin, E. B.: *Markov Processes.* Springer-Verlag, Berlin, 1965

[7] Ethier, S. N. and Kurtz, T. G.: *Markov Processes*: *Characterization and Convergence.* John Wiley and Sons, New York, 1986

[8] Hutzenthaler, M.: Supercritical branching diffusions in random environment. *Elect. Comm. in Probab.* **16**, 781–791 (2011)

[9] Ikeda, N. and Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes.* North Holland, Amsterdam, 1981

[10] Keiding, N.: Extinction and exponential growth in random environments. *Theor. Population Biology* **8**, 49–63 (1975)

[11] Kurtz, T.G.: Diffusion approximations for branching processes. In *Branching Processes, Adv. Prob. 5.* Anatole Joffe and Peter Ney, editors, Marcel Dekker, New York, 262–292 (1978)

[12] Li, Z. H.. *Measure-Valued Branching Markov Processes.* Springer, Heidelberg, 2011

[13] Li, Z. H., Wang, H. and Xiong, J.: Conditional Log-Laplace functionals of immigration superprocesses with dependent spatial motion. *Acta Applicandae Mathematicae* **88(2)**, 143–175 (2005)

[14] Mitoma, I.: Tightness of probabilities on $C([0,1], \mathcal{S}')$ and $D([0,1], \mathcal{S}')$. *Ann. Prob.* **11**, 989–999 (1983)

[15] Mytnik, L.: Superprocesses in random environments. *Ann. Probab.* **24**, 1953–1978 (1996)

[16] Perkins, E. A.: Dawson-Watanabe superprocesses and measure-valued diffusions. *Lecture Notes in Math.* **1781** 125–329. Springer, Berlin, 1993

[17] Protter P.: *Stochstic Intergration and Differential Equations*. Springer, New York, 2004

[18] Smith, W. L. and Wilkinson, W. E.: On branching processes in random environments. *Ann. Math. Statist* **40**, 814–827 (1969)

[19] Wang, H.: State classification for a class of measure-valued branching diffusions in a Brownian medium. *Probab. Th. Rel. Fields* **109**, 39–55 (1997)

[20] Wang, H.: A class of measure-valued branching diffusions in a random medium. *Stochastic Anal. Appl.* **16(4)**, 753–786 (1998)

[21] Xiong, J.: A stochastic log-Laplace equation. *Ann. Probab.* **32(3B)**, 2362–2388 (2004)

[22] Xiong, J.: *Three Classes of Nonlinear Stochastic Partial Differential Equations*. World Scientific Publishing Co. Pte. Ltd. Singapore, 2013