Stochastic Partial Differential Equations for a Class of Interacting Measure-valued Diffusions

D.A. Dawson[*], J. Vaillancourt[†] and H. Wang

The Fields Institute, Université de Sherbrooke and Algorithmics Inc.

Abstract

Using the existence of density processes, we derive a new class of stochastic partial differential equations for a collection of interacting measure-valued diffusions based on two orthogonal martingale measures.

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1 Introduction

Interacting branching measure-valued diffusions (IBMDs) were introduced and characterized by Wang [16, 17] in order to model and study the behavior of one-dimensional super-Brownian motion in a random medium. In Wang [16], it is shown that, when the diffusion coefficient for the medium is smooth enough, these IBMDs either have discrete support or have densities, according to whether or not the differential part of the associated generator is a singular operator. In the latter case, these IBMDs can also be viewed as those superprocesses associated with some of the branching-free interacting diffusion systems of McKean-Vlasov type studied by Kotelenez [11, 12, 13] and by Dawson and Vaillancourt [2].

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[†]Corresponding author: Département de Mathématique et d'Informatique, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1. Research supported by NSERC operating grant.

In the present paper, we derive a new class of stochastic partial differential equations (SPDEs) for the density processes associated with IBMDs, when these densities do exist. In order to state this result precisely, we need the following notation.

We denote by $B(\mathbb{R})$ the space of all bounded Borel measurable functions from $\mathbb{R}$ into itself; $C(\mathbb{R}) \subset B(\mathbb{R})$, its subspace of all bounded continuous functions; $\hat{C}(\mathbb{R}) \subset C(\mathbb{R})$, its subspace of continuous functions which vanish at (infinity) point ∂; $\hat{C}^2(\mathbb{R}) \subset \hat{C}(\mathbb{R})$, its subspace of all twice differentiable functions which vanish at ∂, together with both their first and second derivatives; $\mathcal{S}(\mathbb{R}) \subset \hat{C}^2(\mathbb{R})$, the space of infinitely differentiable functions which, together with all their derivatives, are rapidly decreasing at infinity. We write $\phi'(x)$ or $d\phi(x)/dx$ for the derivative of $\phi \in \hat{C}^2(\mathbb{R})$. $\mathcal{S}'(\mathbb{R})$ is the Schwartz space of tempered distributions and $\mathbf{M}_F(\mathbb{R})$, the Polish space of all bounded Radon measures on $\mathbb{R}$, with the topology of vague convergence. We denote by $< \cdot, \cdot >: \mathcal{S}(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ the usual duality between $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ (and, by extension, that between $B(\mathbb{R})$ and $\mathbf{M}_F(\mathbb{R})$). The variational derivative of $F : \mathbf{M}_F(\mathbb{R}) \to \mathbb{R}$ at μ in direction $z \in \mathbb{R}$ is given (when it exists) by

$$\frac{\delta F(\mu)}{\delta \mu(z)} = \lim_{\epsilon \to 0} (1/\epsilon) \Big(F(\mu + \epsilon \delta_z) - F(\mu) \Big) .$$

Finally, $L^2(\mathbb{R})$ is the usual Lebesgue space of square-integrable functions.

The IBMDs of interest are all the solutions to the martingale problems associated with operators of the form $\mathcal{A} + \mathcal{B}$, with

$$\mathcal{B}F(\mu) := \tfrac{1}{2}\gamma\sigma^2 \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx) \tag{1.1}$$

and

$$\begin{aligned} \mathcal{A}F(\mu) := {} & \tfrac{1}{2} \int_{\mathbb{R}} \rho_\epsilon \frac{d^2}{dx^2} \left(\frac{\delta F(\mu)}{\delta \mu(x)} \right) \mu(dx) \\ & + \tfrac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x-y) \frac{d^2}{dxdy} \left(\frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \right) \mu(dx)\mu(dy) , \end{aligned} \tag{1.2}$$

where, for some given $g \in L^2(\mathbb{R})$, we define the convolution $\rho(z) := \int_{\mathbb{R}} g(z-y)g(y)dy$ and, for convenience, we let $\rho_\epsilon = \rho(0) + \epsilon^2 > 0$ and $\sigma^2 > 0$.

These martingale problems were introduced and studied by Wang [16, 17]. Let us first summarize in Theorem 1.1, three of his results which are relevant here, namely Theorem 2.1 from Wang [16] and Theorems 6.4 and 7.2 from Wang [17].

Theorem 1.1 *Let γ, σ^2 and ϵ be positive constants and let $g \in L^2(\mathbb{R}) \cap \hat{C}(\mathbb{R})$ satisfy $g(-x) = g(x)$ for all $x \in \mathbb{R}$ and be such that $\rho \in \hat{C}^2(\mathbb{R})$ holds. The martingale problem for operator $\mathcal{A} + \mathcal{B}$ described above, started at some measure $\mu_0 \in \mathbf{M}_F(\mathbb{R})$ with compact support, is well-posed. If we write its unique solution as $(\Omega, \mathcal{F}, \mathbb{P}_\mu, \mu_t)$, then the process $\{\mu_t\}$ has a density $\{\ell_t\} \subset L^1(\mathbb{R})$.*

At the present time, no information is available about the regularity in (t, x) of the density $\ell_t(x)$ other than joint measurability. This was established in Wang [16], along with the identity $\int_0^t < \phi^2, \tilde{\mu}_u^0 > du = \int_0^t \int_{\mathbb{R}} \phi^2(x)\ell_u(x)\, dxdu$, which is used in the proof of the following, our main result.

Theorem 1.2 *Let γ, σ^2 and ϵ be positive constants and let $g \in L^2(\mathbb{R}) \cap \hat{C}(\mathbb{R})$ satisfy $g(-x) = g(x)$ for all $x \in \mathbb{R}$ and be such that $\rho \in \hat{C}^2(\mathbb{R})$ holds. Let $(\Omega, \mathcal{F}, \mathbb{P}_\mu, \mu_t)$ be the IBMD which is the solution to the well-posed martingale problem in Theorem 1.1 above. There exist two orthogonal (hence independent) $\mathcal{S}'(\mathbb{R})$-valued cylindrical Brownian motions V_t and W_t defined on an extended probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}_\mu)$ of the original space $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ such that for every $\phi \in \mathcal{S}(\mathbb{R})$,*

$$(1.3)\quad < \phi, \mu_t > - < \phi, \mu_0 > -\frac{1}{2}\rho_\epsilon \int_0^t < \phi'', \mu_s > ds$$
$$= \int_0^t \int_{\mathbb{R}} \sqrt{\gamma\sigma^2\ell_s(x)}\phi(x)V(dx, ds) + \int_0^t \int_{\mathbb{R}} < g(y-x)\phi'(x), \mu_s(dx) > W(dy, ds)$$

holds for every $t \geq 0$, $\bar{\mathbb{P}}_\mu$-almost surely.

The derivation of this SPDE runs roughly as follows: we first derive a Quasi-SPDE for the sequence of empirical measure-valued processes associated with the generating, finite particle systems, by way of the strong construction of a copy of the whole sequence on a common probability space; we then prove the tightness and L^p-convergence of each term in the Quasi-SPDE; finally the solution to the SPDE of Theorem 1.2 emerges from the Quasi-SPDE by letting the size of the system grow to infinity.

The special case of Theorem 1.2 obtained by assuming $g \equiv 0$ was the subject of a seminal paper of Konno and Shiga [10], where the independence of motion of the particles at every level allowed them to use a representation theorem for individual martingale measures. As one sees upon glancing at equation (1.3), the strong dependence between the motion of the various particles (even amongst the infinite system) gives rise to not one but two martingale measures, which turn out to be orthogonal to each other. The original approach of Konno and Shiga is therefore not directly applicable here.

Our derivation is analogous to that in Kotelenez [12, 13], where the sources of motion for all the finite systems involved are a (deterministic) free force field and a highly correlated random environment. In the present paper, however, serious mathematical difficulties are introduced by allowing the particles to execute branching Brownian motions, independently of one another, given the state of the random environment — there is no force field here. The explicit construction provided in Section 2 gets us around these difficulties; the rest of the argument relies crucially on a decomposition theorem for orthogonal martingale measures. This approach for the derivation of the SPDE for empirical measure-valued processes is inspired in part by the pioneering work of Walsh [15] and extends some of his results to systems of highly dependent particles.

2 Proofs

The proof of the main result Theorem 1.2 is achieved by way of a series of lemmata. Let us begin by building explicitly a version of the model for IBMDs discussed in Wang [17]. The evolution of the particle system in between branching times takes the form of a strong solution to the following stochastic evolution equation.

Lemma 2.1 *Given $\epsilon \neq 0$, let $g \in L^2(\mathbb{R}) \cap \hat{C}(\mathbb{R})$ satisfy $g(-x) = g(x)$ for all $x \in \mathbb{R}$ and be such that $\rho \in \hat{C}^2(\mathbb{R})$ holds. Let W be a cylindrical Brownian motion and $\{B^\alpha\}$ a countable collection of standard one-dimensional Brownian motions, built on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of each other. Then the following system of stochastic integral equations*

$$z_t^\alpha - z_0^\alpha = \int_0^t \int_{\mathbb{R}} g(y - z_s^\alpha) W(dy, ds) + \epsilon \, B_t^\alpha \tag{2.4}$$

has a unique strong solution with continuous paths, for every countable collection of starting points $\{z_0^\alpha\} \subset \mathbb{R}$.

Proof: The usual method of successive approximations works here — see Wang [17] or Kotelenez [12, 13] for more details. ■

Since the strong solution of (2.4) only depends on the initial state z_0^α, the process $B^\alpha = \{B_t^\alpha : t \geq 0\}$ and a common W, we denote it by $z_t^\alpha = x(z_0^\alpha, B^\alpha, t)$ for some measurable real-valued mapping x (omitting W in the notation as it is selected and fixed once and for all). We note in passing that, by Itô's formula, for any $\phi \in \mathcal{S}(\mathbb{R})$, the unique solution to equation (2.4) verifies

$$\begin{aligned} \phi(z_t^\alpha) - \phi(z_0^\alpha) &= \int_0^t \int_{\mathbb{R}} \phi'(z_s^\alpha) g(y - z_s^\alpha) W(dy, ds) \\ &\quad + \epsilon \int_0^t \phi'(z_s^\alpha) dB_s^\alpha + \frac{\rho_\epsilon}{2} \int_0^t \phi''(z_s^\alpha) ds. \end{aligned} \tag{2.5}$$

Using this infinite collection of solutions to equation (2.4), we can now build the IBMDs as weak limits of a whole sequence of finite particle systems, denoted by $\{x^\alpha\}$ and all built on a common probability space, as follows.

For any positive integer n, there is an initial system of m_0^n particles, each particle having mass $1/\theta^n$ and branching at rate $\gamma\theta^n$. The offspring distribution $\{p_k\}$ satisfies

$$p_1 = 0, \qquad \sum_{k=0}^\infty k p_k = 1, \qquad \sum_{k=0}^\infty k^2 p_k < \infty,$$

and $m_0^n/\theta^n \le \xi$, where both $\xi > 0$ and $\theta \ge 2$ are fixed constants.

Let $\Re$ be the set of all multi-indices, i.e., strings of the form $\alpha = n_1 n_2 \cdots n_k$, where the n_i's are non-negative integers. Let $|\alpha|$ be the length of α. We provide $\Re$ with the arboreal ordering: $m_1 m_2 \cdots m_p \prec n_1 n_2 \cdots n_q$ iff $p \le q$ and $m_1 = n_1, \cdots, m_p = n_p$. If $|\alpha| = p$, then α has exactly $p-1$ predecessors, which we shall denote respectively by $\alpha-1, \alpha-2, \cdots, \alpha-|\alpha|+1$. For example, with $\alpha = 6879$, we get $\alpha-1 = 687$, $\alpha-2 = 68$ and $\alpha-3 = 6$.

Define three independent families $\{B^\alpha, \alpha \in \Re\}$, $\{S^\alpha, \alpha \in \Re\}$ and $\{N^\alpha, \alpha \in \Re\}$, where the B^α's are independent standard Brownian motions in $\mathbb{R}$; the S^α's are i.i.d. exponential random variables with parameter $\gamma\theta^n$, which serve as lifetimes; and the N^α's are i.i.d. random variables with $\mathbb{P}(N^\alpha = k) = p_k$ for $k = 0, 1, 2, \cdots$ and $p_1 = 0$.

The birth time $\beta(\alpha)$ of x^α is defined by

$$\beta(\alpha) := \begin{cases} \sum_{j=1}^{|\alpha|-1} S^{\alpha-j} & \text{if } N^{\alpha-j} \ge 2 \text{ holds for every } j = 1, \cdots, |\alpha|-1\,; \\ \infty & \text{otherwise.} \end{cases}$$

The death time of x^α is defined by $\zeta(\alpha) = \beta(\alpha) + S^\alpha$ and the indicator function of the lifespan of x^α is denoted by $h^\alpha(t) = 1_{\{\beta(\alpha) \le t < \zeta(\alpha)\}}$.

Recall that ∂ denotes a point at infinity — the cemetery — and put $x_t^\alpha = \partial$ if either $t < \beta(\alpha)$ or $t \ge \zeta(\alpha)$ holds. We make the convention that any function f on $\mathbb{R}$ is extended to $\mathbb{R} \cup \{\partial\}$ by setting $f(\partial) = 0$ — this allows us to keep track of only those particles not in the cemetery at any given time.

Given $\mu_0 \in \mathbf{M}_F(\mathbb{R})$ with compact support, assume that $\mu_0^n = (1/\theta^n) \sum_{\alpha=1}^{m_0^n} \delta_{x_0^\alpha}$ is constructed from μ_0 as in Wang [17] so that $\mu_0^n \Rightarrow \mu_0$ holds as $n \to \infty$. We are thus provided with collections of starting positions $\{x_0^\alpha\}$ for each $n \ge 1$.

Let $\mathcal{N}_1^n := \{1, 2, \cdots, m_0^n\}$ be the set of indices for the first generation of particles. For any $\alpha \in \mathcal{N}_1^n \cap \Re$, define

$$x^\alpha(t) = \begin{cases} x(x_0^\alpha, B^\alpha, t) & 0 \le t < S^\alpha \\ \partial & S^\alpha \le t \end{cases} \tag{2.6}$$

and

$$x^\alpha(t) \equiv \partial \qquad \text{for any } \alpha \in (\mathbb{N} - \mathcal{N}_1^n) \cap \Re \text{ and } t \ge 0.$$

For the path of the second generation, let $\bar{\zeta}_1 = \min\{S^\alpha : \alpha \in \mathcal{N}_1^n \cap \Re\}$. By Ikeda-Nagasawa-Watanabe [6], for each $\omega \in \Omega$ there exists a measurable selection $\alpha_0 = \alpha_0(\omega) \in \mathcal{N}_1^n \cap \Re$ such that $\bar{\zeta}_1(\omega) = S^{\alpha_0}(\omega)$.

If $N^{\alpha_0}(\omega) = k \ge 2$, define, for every $\alpha \in \{\alpha_0 i; i = 1, 2, \cdots, k\}$,

$$x^\alpha(t) = \begin{cases} x(x^{\alpha_0}(\zeta(\alpha_0)-), B^\alpha, t) & \beta(\alpha) \le t < \zeta(\alpha) \\ \partial & \text{otherwise.} \end{cases} \tag{2.7}$$

If $N^{\alpha_0}(\omega) = 0$, define $x^\alpha(t) = \partial$ for $0 \le t < \infty$ and $\alpha \in \{\alpha_0 i : i \ge 1\}$.

More generally, suppose there have been m splits already. Let $\mathcal{N}_m^n \subset \Re$ be the set of all indices for the living particles and let $\bar\zeta_{m+1} = \min\{S^\alpha : \alpha \in \mathcal{N}_m^n\}$. Then for each ω, there exists $\beta_0 \in \mathcal{N}_m^n$ such that $\bar\zeta_{m+1}(\omega) = S^{\beta_0}(\omega)$. If $N^{\beta_0}(\omega) = k \ge 2$, define

$$x^\alpha(t) = \begin{cases} x(x^{\beta_0}(\zeta(\beta_0)-), B^\alpha, t) & \beta(\alpha) \le t < \zeta(\alpha) \\ \partial & \text{otherwise} \end{cases} \tag{2.8}$$

for $\alpha \in \{\beta_0 i; i = 1, 2, \cdots, k\}$. If $N^{\beta_0}(\omega) = 0$, define

$$x^\alpha(t) = \partial \quad \text{for} \quad 0 \le t < \infty \quad \text{and} \quad \alpha \in \{\beta_0 i : i \ge 1\}.$$

Continuing in this way, we get a branching tree of particles for any given ω with initial state selected at random amongst $\{x_0^1, x_0^2, \cdots, x_0^{m_0^n}\}$.

Define the associated empirical process

$$\mu_t^n := \frac{1}{\theta^n} \sum_{\alpha \in \Re} \delta_{x^\alpha(t)}. \tag{2.9}$$

For any $A \in \mathcal{B}(\mathbb{R})$ and $t > 0$, define what will turn out to be an approximation "à la Donsker" to a new cylindrical Brownian motion:

$$Z^n(A \times (0,t]) := \sum_{\alpha \in \Re} \frac{(N^\alpha - 1)}{\theta^n} 1_{\{x^\alpha(\zeta(\alpha)-) \in A, \zeta(\alpha) \le t\}}, \tag{2.10}$$

representing a (scaled down) "brood size" for those particles dead by time t and positioned inside A upon the advent of their demise.

Observation (2.5) above implies that each μ^n satisfies, for every $\phi \in \mathcal{S}(\mathbb{R})$,

$$< \phi, \mu_t^n > - < \phi, \mu_0^n > = \frac{1}{\sqrt{\theta^n}} U_t^n(\phi) + X_t^n(\phi) + Y_t^n(\phi) + Z_t^n(\phi), \tag{2.11}$$

where we use the notation

$$U_t^n(\phi) := \frac{\epsilon}{\sqrt{\theta^n}} \sum_{\alpha \in \Re} \int_0^t h^\alpha(s) \phi'(x^\alpha(s)) dB_s^\alpha,$$

$$X_t^n(\phi) := \int_0^t \int_{\mathbb{R}} < g(y - \cdot)\phi'(\cdot), \mu_s^n > W(dy, ds),$$

$$Y_t^n(\phi) := \frac{\rho_\epsilon}{2} \int_0^t < \phi'', \mu_s^n > ds,$$

$$Z_t^n(\phi) := \int_0^t \int_{\mathbb{R}} \phi(x) Z^n(dx, ds) \,.$$

The four terms represent the respective components of the overall motion of the finite particle systems $\mu_t^n(\phi) :=< \phi, \mu_t^n >$ contributed by the individual Brownian motions $(U_t^n(\phi))$, the random medium $(X_t^n(\phi))$, their common diffusive effect $(Y_t^n(\phi))$ and the branching mechanism $(Z_t^n(\phi))$. Using a result of Dynkin ([4] p.325, Theorem 10.13), we get at once the following theorem.

Theorem 2.2 *$\forall n \in \mathbb{N}$, μ_t^n defined by* (2.9) *is a right continuous strong Markov process which is a unique strong solution of* (2.11) *in the sense that it is a unique solution of* (2.11) *for fixed probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and given* W, $\{B^\alpha\}$, $\{S^\alpha\}$, $\{N^\alpha\}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Furthermore, all the* $\{\mu_t^n; t \geq 0\}$ *are defined on the common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Denote by $\{\mathcal{F}_t^n : t \geq 0\}$ the filtration defined by writing $\mathcal{F}_t^n$ for the σ-algebra generated by the collection of processes

$$\Big\{ \mu_u^n(\phi), U_u^n(\phi), X_u^n(\phi), Y_u^n(\phi), Z_u^n(\phi), \int_0^u h(\mu_s^n(\phi))ds, \int_0^u h(Z_s^n(\phi))ds, \int_0^u h(U_s^n(\phi))ds, \int_0^u h(X_s^n(\phi))ds, \int_0^u h(Y_s^n(\phi))ds \,; \, 0 \leq u \leq t, \phi \in \mathcal{S}(\mathbb{R}), h \in B(\mathbb{R}) \Big\} \,.$$

Lemma 2.3 *If we write* $\sigma^2 := (\sum_{k=0}^\infty k^2 p_k) - 1$, *we have*
(i) *For every* $\phi \in \mathcal{S}(\mathbb{R})$, $\mathbb{E} Z_t^n(\phi)^2 = \gamma \sigma^2 \mathbb{E} \int_0^t < \phi^2, \mu_u^n > du$;
(ii) *For any* $T \geq 0$ *and* $n \geq 1$, *we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} < 1, \mu_t^n > \;\leq\; < 1, \mu_0^n > + \; 16 \sqrt{\gamma \sigma^2 T < 1, \mu_0^n >} \;\; \textit{and}$$

$$\mathbb{E} \sup_{0 \leq t \leq T} < 1, \mu_t^n >^2 \;\leq\; 2 < 1, \mu_0^n >^2 + \; 8 \gamma \sigma^2 T < 1, \mu_0^n > \,.$$

(iii) $\{\mu_t^n; t \geq 0\}$ *defined by* (2.9) *is tight as a family of processes with sample paths in* $D([0, \infty), M_F(\mathbb{R}))$.

Proof: (i) Remembering that $\{S^\alpha, \alpha \in \Re\}$ are i.i.d. exponential random variables with parameter $\gamma \theta^n$ and $h^\alpha(t) = 1_{\{\beta(\alpha) \leq t < \zeta(\alpha)\}}$, for any $A \in \mathcal{B}(\mathbb{R})$, we have

$$\mathbb{E}\{1_{(s,t]}(\zeta(\alpha)) 1_{(x^\alpha(\zeta(\alpha)-) \in A)}\} = \mathbb{E}\left\{ \int_s^t h^\alpha(u) 1_{(x^\alpha(u) \in A)} \gamma \theta^n du \right\} \,.$$

Therefore, we get

$$
\begin{aligned}
(2.12)\qquad \mathbb{E}Z_t^n(\phi)^2 &= \mathbb{E}\sum_{\alpha\in\Re}\frac{(N^\alpha-1)^2}{\theta^{2n}}\phi^2(x^\alpha(\zeta(\alpha)-))1_{(\zeta(\alpha)\le t)}\\
&= \gamma\sigma^2\frac{1}{\theta^n}\mathbb{E}\sum_{\alpha\in\Re}\int_0^t h^\alpha(u)\phi^2(x^\alpha(u))du\\
&= \gamma\sigma^2\mathbb{E}\int_0^t <\phi^2,\mu_u^n> du\,.
\end{aligned}
$$

(ii) Since $<1,\mu_t^n-\mu_0^n>=Z_t^n(1)$ is a zero-mean martingale, by Davis's inequality (see Dellacherie-Meyer [3]) we have

$$
\mathbb{E}\sup_{0\le t\le T}|<1,\mu_t^n-\mu_0^n>|\le 16\sqrt{\gamma\sigma^2\mathbb{E}\int_0^T <1,\mu_u^n> du} = 16\sqrt{\gamma\sigma^2 T<1,\mu_0^n>}\,.
$$

Similarly, Doob's submartingale inequality yields

$$
\mathbb{E}\sup_{0\le t\le T}<1,\mu_t^n-\mu_0^n>^2\le\ 4\gamma\sigma^2T<1,\mu_0^n>\ .
$$

(iii) By Theorems 4.5.4 and 4.6.1 in Dawson [1] (plus part (ii), which precludes explosion in finite time), we only need to prove that, for any given $\epsilon>0$, $T>0$, $\phi\in\mathcal{S}(\mathbb{R})$ and any stopping time τ_n bounded by T, then $\forall\,\eta>0$, $\exists\,\delta,n_0$ such that $\sup_{n\ge n_0}\sup_{t\in[0,\delta]}\mathbb{P}\{|\mu^n_{\tau_n+t}(\phi)-\mu^n_{\tau_n}(\phi)|>\epsilon\}\le\eta$.

Using the strong Markov property of $\mu^n(\phi)$, we obtain from (i)

$$
(2.13)\quad \mathbb{P}(|\mu^n_{\tau_n+t}(\phi)-\mu^n_{\tau_n}(\phi)|>\epsilon)=\mathbb{P}(|\mu^n_t(\phi)-\mu^n_0(\phi)|>\epsilon)
$$

$$
\le\frac{4}{\epsilon^2}\left\{\mathbb{E}X_t^n(\phi)^2+\mathbb{E}Y_t^n(\phi)^2+\frac{1}{\theta^n}\mathbb{E}U_t^n(\phi)^2+\mathbb{E}Z_t^n(\phi)^2\right\}
$$

$$
\le\frac{4}{\epsilon^2}\left\{\mathbb{E}\int_0^t <\rho(x-y)\phi'(x)\phi'(y),\mu_s^n(dx)\mu_s^n(dy)> ds\right.
$$

$$
\left.+\frac{\rho_\epsilon^2 t}{4}\mathbb{E}\int_0^t <\phi'',\mu_s^n>^2 ds+\frac{\epsilon^2}{\theta^n}\mathbb{E}\int_0^t <(\phi')^2,\mu_s^n> ds\,+\,\gamma\sigma^2\mathbb{E}\int_0^t <\phi^2,\mu_u^n> du\right\}\,,
$$

which goes to 0 as $t\to 0$. ■

Lemma 2.4 (i) (μ^n, Z^n, U^n, Y^n) *is tight on* $D([0,\infty), (\mathcal{S}'(\mathbb{R}))^4)$.
(ii) *(A Skorohod representation): Suppose the joint distribution of*

$$(\mu^{n_k}, Z^{n_k}, U^{n_k}, Y^{n_k}, W) \Longrightarrow (\mu^0, Z^0, U^0, Y^0, W)$$

converges weakly, then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *and* $D([0,\infty), \mathcal{S}'(\mathbb{R}))$*-valued sequences* $\{\tilde{\mu}^{n_k}\}$, $\{\tilde{Z}^{n_k}\}$, $\{\tilde{U}^{n_k}\}$, $\{\tilde{Y}^{n_k}\}$, *and* $\{\tilde{W}^{n_k}\}$ *defined on it, such that* $\mathbb{P} \circ (Z^{n_k}, \mu^{n_k}, U^{n_k}, Y^{n_k}, W)^{-1} = \tilde{\mathbb{P}} \circ (\tilde{Z}^{n_k}, \tilde{\mu}^{n_k}, \tilde{U}^{n_k}, \tilde{Y}^{n_k}, \tilde{W}^{n_k})^{-1}$ *holds and,* $\tilde{\mathbb{P}}$*-almost surely on* $D([0,\infty), (\mathcal{S}'(\mathbb{R}))^5)$, $(\tilde{Z}^{n_k}, \tilde{\mu}^{n_k}, \tilde{U}^{n_k}, \tilde{Y}^{n_k}, \tilde{W}^{n_k}) \to (\tilde{Z}^0, \tilde{\mu}^0, \tilde{U}^0, \tilde{Y}^0, \tilde{W}^0)$.
(iii) $(\tilde{Z}^{n_k}(\phi), \tilde{\mu}^{n_k}(\phi), \tilde{U}^{n_k}(\phi), \tilde{Y}^{n_k}(\phi)\tilde{W}^{n_k}(\phi)) \to (\tilde{Z}^0(\phi), \tilde{\mu}^0(\phi), \tilde{U}^0(\phi), \tilde{Y}^0(\phi), \tilde{W}^0(\phi))$ *in* $L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *as an* $\mathbb{R}^5$*-valued process, for any* $\phi \in \mathcal{S}(\mathbb{R})$.
(iv) $\tilde{W}^0(dy, ds)$ *and* $\tilde{W}^{n_k}(dy, ds)$ *are cylindrical Brownian motions and the following stochastic integrals converge*

$$\tilde{X}_t^{n_k}(\phi) := \int_0^t \int_{\mathbb{R}} < g(y-\cdot)\phi'(\cdot), \tilde{\mu}_s^{n_k} > \tilde{W}^{n_k}(dy, ds)$$
$$\to \tilde{X}_t^0(\phi) := \int_0^t \int_{\mathbb{R}} < g(y-\cdot)\phi'(\cdot), \tilde{\mu}_s^0 > \tilde{W}^0(dy, ds) \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}).$$

(v) $(\tilde{\mu}^0, \tilde{Z}^0, \tilde{X}^0)$ *is unique in distribution and satisfies the equation*

$$(2.14)\,\tilde{\mu}_t^0(\phi) - \tilde{\mu}_0^0(\phi) = \tilde{X}_t^0(\phi) + \frac{\rho_\epsilon}{2}\int_0^t < \phi'', \tilde{\mu}_s^0 > ds + \int_0^t \int_{\mathbb{R}} \phi(x)\tilde{Z}^0(dx, ds)\,.$$

(vi) $\tilde{Z}^{n_k}$ *is orthogonal to* $\tilde{X}^{n_k}$ *and* $\tilde{Z}^0$ *is orthogonal to* $\tilde{X}^0$.

Proof: (i) By a theorem of Mitoma [14], we only need to prove that, for any $\phi \in \mathcal{S}(\mathbb{R})$, the sequence of laws for $(\mu^n(\phi), Z^n(\phi), U^n(\phi), Y^n(\phi))$ is tight in $D([0,\infty), \mathbb{R}^4)$. This is equivalent to proving that each component and the sum of each pair of components are individually tight in $D([0,\infty), \mathbb{R})$. Since the same idea works for each sequence, we only give the proof for $\{Z^n(\phi)\}$. Let $C = \sup_x \phi(x)^2$ and use Lemma 2.3 to get

$$\mathbb{P}\{Z_t^n(\phi) > n\} \le \frac{\gamma\sigma^2}{n^2}\mathbb{E}\int_0^t < \phi^2, \mu_u^n > du \le \frac{\gamma\sigma^2 Ct}{n^2} < 1, \mu_0^n >,$$

which yields the compact containment condition. Now we use Kurtz's tightness criterion (cf. Ethier-Kurtz [5] p. 137, Theorem 8.6) to prove the tightness of $\{Z^n(\phi)\}$.

Let $\gamma_n^T(\delta) := \delta\gamma\sigma^2 C \sup_{0\le u\le T} < 1, \mu_u^n >$, then for any $0 \le t+\delta \le T$,

$$\mathbb{E}\{|Z_{t+\delta}^n(\phi) - Z_t^n(\phi)|^2 | \mathcal{F}_t^n\} = \mathbb{E}\{\gamma\sigma^2 \int_t^{t+\delta} < \phi^2, \mu_u^n > du | \mathcal{F}_t^n\} \le \mathbb{E}\{\gamma_n^T(\delta)|\mathcal{F}_t^n\}\,.$$

By Lemma 2.3, $\lim_{\delta\to 0}\sup_n \mathbb{E}\{\gamma_n^T(\delta)\} = 0$ holds, so $\{Z^n(\phi)\}$ is tight.

(ii) If we choose any countable dense subset $\{g_i\}_{i\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R})$ and any enumeration $\{t_j\}_{j\in\mathbb{N}}$ of all rational numbers, then Theorem 1.7 of Jakubowski [7] shows that the countable family $\{f_{ij} : i, j \in \mathbb{N}\}$ of continuous functions (with respect to Skorohod topology on $D([0,\infty), \mathcal{S}'(\mathbb{R})))$ separates points, when we define $f_{ij} : D([0,\infty), \mathcal{S}'(\mathbb{R})) \to [-\pi, \pi]$ by $f_{ij}(x) = \arctan < g_i, x(t_j) >$. This proves that space $D([0,\infty), \mathcal{S}'(\mathbb{R}))$ — and by an easy extension space $D([0,\infty), (\mathcal{S}'(\mathbb{R}))^5)$ — verifies the basic assumption for a version of the Skorohod Representation Theorem due to Jakubowski [9].

(iii) From Lemma 2.3, given any $\phi \in \mathcal{S}(\mathbb{R})$, we obtain the uniform integrability of $\tilde{\mu}^{n_k}(\phi)$, $\tilde{Z}^{n_k}(\phi)$, $\tilde{U}^{n_k}(\phi)$, $\tilde{Y}^{n_k}(\phi)$ and $(\tilde{Z}^{n_k}(\phi), \tilde{\mu}^{n_k}(\phi), \tilde{W}^{n_k}(\phi))$. So (ii) implies (iii).

(iv) Since W, $\tilde{W}^0$ and $\tilde{W}^{n_k}$ have the same distribution, $\tilde{W}^0$ and $\tilde{W}^{n_k}$ are cylindrical Brownian motions. In view of the continuous embedding of $M_F(\mathbb{R})$ into the Sobolev dual space H_{-3} (see Dawson-Vaillancourt [2] Proposition 5.1), the conclusion follows from Lemma 2.3 (which yields tightness), Lemma 2.4 (iii) (which guarantees the uniqueness of the limit) and Jakubowski's results on the continuity of the Itô stochastic integral in Hilbert spaces (see [8]).

(v) Since

$$\frac{\epsilon}{\theta^n} \sum_{\alpha \in \Re} \int_0^t h^\alpha(s)\phi'(x^\alpha(s))dB_s^\alpha = \frac{1}{\sqrt{\theta^n}} \tilde{U}_t^n(\phi) \to 0 \quad \text{a.s. } (\tilde{\mathbb{P}})$$

holds, we get (2.14) by way of

$$\begin{aligned} 0 &= \lim_{k\to\infty} \tilde{\mathbb{E}} |\tilde{\mu}_t^{n_k}(\phi) - \tilde{\mu}_0^{n_k}(\phi) - \tilde{X}_t^{n_k}(\phi) - \tilde{Y}_t^{n_k}(\phi) - \frac{1}{\sqrt{\theta^n}} \tilde{U}_t^{n_k}(\phi) - \tilde{Z}_t^{n_k}(\phi)| \\ &= \tilde{\mathbb{E}} |\tilde{\mu}_t^0(\phi) - \tilde{\mu}_0^0(\phi) - \tilde{X}_t^0(\phi) - \tilde{Y}_t^0(\phi) - \tilde{Z}_t^0(\phi)| \quad \phi \in \mathcal{S}(\mathbb{R}), \forall t \geq 0. \end{aligned}$$

By Itô's formula, we see that $\{\tilde{\mu}_t^0; t \geq 0\}$ is a solution to the martingale problem for $(\mathcal{A} + \mathcal{B}, \delta_{\mu_0})$. The uniqueness of $\{\tilde{\mu}_t^0; t \geq 0\}$ follows from Theorem 4.1 of Wang [17]. This also implies that $\tilde{X}^0 + \tilde{Z}^0$ is unique in distribution. From (iv), the uniqueness of $\tilde{X}^0$ is obvious. Combining these facts, we reach the conclusion.

(vi) Since $\tilde{Z}^{n_k}$ is a purely discontinuous martingale while $\tilde{X}^{n_k}$ is a continuous martingale, they are orthogonal — see Theorem 43 in Dellacherie-Meyer [3] p. 353. From Corollary 7.3 of Wang [17], we have

$$< \tilde{X}^0(\phi) + \tilde{Z}^0(\phi) >_t = \gamma\sigma^2 \int_0^t < \phi^2, \tilde{\mu}_u^0 > du + \int_0^t \int_{\mathbb{R}} < g(y - \cdot)\phi'(\cdot), \tilde{\mu}_s^0 >^2 dyds \,.$$

So there holds $< \tilde{X}^0(\phi), \tilde{Z}^0(\phi) >_t = 0$ $\mathbb{P}$-almost surely $\forall\, t \geq 0$. This implies the orthogonality of $\tilde{Z}^0$ and $\tilde{X}^0$. ■

We now turn to the proof of Theorem 1.2. ¿From Lemma 2.3 and Lemma 2.4, we have

$$< \int_0^t \int_{\mathbb{R}} \phi(x)\tilde{Z}^0(dx, ds) > = \gamma\sigma^2 \int_0^t < \phi^2, \tilde{\mu}_u^0 > du \,.$$

Now let $\overline{V}$ be an $\mathcal{S}'(\mathbb{R})$-valued cylindrical Brownian motion which is independent of μ and $\tilde{W}^0$ (and hence also of $\tilde{X}^0$) — if necessary, we construct $\overline{V}$ on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}_\mu)$ of probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_\mu)$). Let $\ell_s(x)$ be the density process of $\tilde{\mu}_s^0$ constructed in Wang [16]. Set

$$\begin{aligned} V_t(\phi) = & \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{\gamma\sigma^2 \ell_s(x)}} 1_{(\ell_s(x)\neq 0)} \phi(x) \tilde{Z}^0(dx, ds) \\ & + \int_0^t \int_{\mathbb{R}} 1_{(\ell_s(x)=0)} \phi(x) \overline{V}(dx, ds)\,, \end{aligned}$$

by restricting the first spatial integral on the right hand side to $\{x : n^{-1} \leq |\ell_s(x)| \leq n\}$ and then letting $n \uparrow \infty$. Then it is easy to verify that V_t is an $\mathcal{S}'(\mathbb{R})$-valued cylindrical Brownian motion and that $\tilde{Z}_t^0(\phi) = \int_0^t \int_{\mathbb{R}} \sqrt{\gamma\sigma^2 \ell_s(x)}\, \phi(x) V(dx, ds)$. This last expression, Lemma 2.4 (vi) and the definition of V, together with the independence of $\overline{V}$ and $\tilde{W}^0$ stated above, imply the independence of V and $\tilde{W}^0$ (and hence that of V and W in the statement of the Theorem). Note that all the terms in (2.11) converge in $L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_\mu)$. Taking limits in (2.11) and using Lemma 2.4, we get the desired SPDE (1.3) for $\tilde{\mu}_t^0$. ■

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Titre en français : Équations aux dérivées partielles stochastiques pour une classe de diffusions interactives à valeurs mesures

Résumé : Nous introduisons une nouvelle classe d'équations aux dérivées partielles stochastiques, engendrées par deux mesures martingales orthogonales, pour caractériser une famille de diffusions à valeurs mesures avec interactions, en exploitant l'existence de densités.

Donald A. DAWSON
The Fields Institute
222 College Street, 2nd Floor

Toronto (On) Canada M5T 3J1

Jean VAILLANCOURT
Faculté des sciences
Université de Sherbrooke
Sherbrooke (Qc) Canada J1K 2R1

Hao WANG
Algorithmics Inc.
185 Spadina Ave,
Toronto (On) Canada M5T 2C6