Completing the square

1 Why do we need to know this?

Completing the square can be used to solve quadratics (as an alternative to the quadratic formula). However, most students find the quadratic formula easier. Why learn how to complete the square then? Completing the square is used in techniques we will cover later, such as identifying circles, or graph more complicated functions—situations in which the quadratic formula does not apply. Thus, be sure you understand how to complete the square **now** (if you haven't already)!

2 Why and how does this work?

1. We can use FOIL to find that
$$\left(x+\left(\frac{b}{2}\right)\right)^2 = x^2 + \frac{2b}{2}x + \left(\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2.$$
   So, we can flip things around and we get
$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x+\left(\frac{b}{2}\right)\right)^2.$$
2. Thus, **whenever we get anything in the form** $x^2 + bx + \left(\frac{b}{2}\right)^2$, we can replace it with $\left(x+\left(\frac{b}{2}\right)\right)^2$.
3. If we have something in the form
$$x^2 + bx = t$$
   we can add $\left(\frac{b}{2}\right)^2$ to both sides, so we get
$$x^2 + bx + \left(\frac{b}{2}\right)^2 = t + \left(\frac{b}{2}\right)^2.$$
   It is this step that we call *completing the square*, since by adding $\left(\frac{b}{2}\right)^2$ to both sides, we have made the left side a complete square. Now, we can replace the left side and get
$$\left(x+\frac{b}{2}\right)^2 = t + \left(\frac{b}{2}\right)^2.$$
4. Then we can take the square root of both sides, and get
$$x + \frac{b}{2} = \pm\sqrt{t + \left(\frac{b}{2}\right)^2},$$
   and solve from there.

3 The process in a nutshell

1. Divide out the x^2 coefficient.
2. Move constant to the right side.
3. Add $\left(\frac{b}{2}\right)^2$ to both sides (complete the square).
4. You can now factor left side.
5. Take square root.
6. Enjoy! (Enjoy solving, that is.)

4 An example

Here is an example:

$$2x^2 - 5x - 3 = 0$$

$$x^2 - \tfrac{5}{2}x - \tfrac{3}{2} = 0 \qquad \text{(Step 1, divide by } x^2 \text{ coefficient)}$$

$$x^2 - \tfrac{5}{2}x = \tfrac{3}{2} \qquad \text{(Step 2, Move constant to right side)}$$

Side arithmetic:

$$b = \tfrac{5}{2}, \text{so } \tfrac{b}{2} = \tfrac{5}{2} \cdot \tfrac{1}{2} = \tfrac{5}{4},$$

$$\text{so } \left(\tfrac{b}{2}\right)^2 = \left(\tfrac{5}{4}\right)^2 = \tfrac{25}{16}$$

$$\text{Also, } \tfrac{3}{2} = \tfrac{24}{16}$$

$$x^2 - \tfrac{5}{2}x + \tfrac{25}{26} = \tfrac{24}{16} + \tfrac{25}{26} \qquad \text{Step 3, Add } \left(\tfrac{b}{2}\right)^2 \text{ to both sides.}$$

$$\left(x - \tfrac{5}{4}\right)^2 = \tfrac{49}{16} \qquad \text{(Step 4, Factor left side)}$$

$$x - \tfrac{5}{4} = \pm\sqrt{\tfrac{49}{16}} \qquad \text{(Step 5, Take square root)}$$

$$x - \tfrac{5}{4} = \pm\tfrac{7}{4}$$

$$x = \tfrac{5}{4} \pm \tfrac{7}{4}$$

$$x = \tfrac{5}{4} + \tfrac{7}{4} \text{ or } x = \tfrac{5}{4} - \tfrac{7}{4}$$

$$x = -\tfrac{1}{2} \text{ or } x = 3 \qquad \text{(Step 6, Enjoy solving, because it's easier.)}$$