Multiplicity-free subgroups of reductive algebraic groups

Jonathan Brundan

Abstract

We introduce the notion of a multiplicity-free subgroup of a reductive algebraic group in arbitrary characteristic. This concept already exists in the work of Krämer for compact connected Lie groups. We give a classification of reductive multiplicity-free subgroups, and as a consequence obtain a simple proof of a theorem of Kleshchev.

1 Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$. If $p = 0$, it is well known that the restriction of any irreducible $SL_n(k)$-module to the natural subgroup $GL_{n-1}(k)$ is multiplicity-free. The same is true for the restriction of an irreducible $SO_n(k)$-module to the subgroup $SO_{n-1}(k)$. In positive characteristic, these results are no longer true, but a recent result of Kleshchev [8, Theorem A] shows nonetheless that the socle and the head (which is isomorphic to the socle) of the restriction are both multiplicity-free. In our first theorem, we give a simple proof of this fact, which is quite different from Kleshchev's original proof.

Theorem A. *Let $H < G$ be the simply connected cover of an entry in table 1. Then*

$$\dim \mathrm{Hom}_H(\Delta_H, \nabla_G) \leq 1 \qquad (1)$$

for all Weyl modules Δ_H for H and all coWeyl modules ∇_G for G. Hence, the socle and head of $\mathrm{res}^G_H L_G$ are multiplicity-free for every irreducible G-module L_G.

Kleshchev actually proves a slightly weaker result, namely that for all pairs (G, H) in Theorem A, $\dim \mathrm{Hom}_H(\Delta_H, L_G) \leq 1$ for all Weyl modules Δ_H for H and all irreducible modules L_G for G. Kleshchev also shows how to deduce from this result in the case $(G, H) = (SL_n(k), GL_{n-1}(k))$ (by applying a "Schur functor") that the restriction of any irreducible module for the symmetric group algebra $k\mathfrak{S}_r$ to $k\mathfrak{S}_{r-1}$ has multiplicity-free socle and head. In later work [9, 10], Kleshchev has described precisely

Table 1: Multiplicity-free subgroups

G		H
$SL_n(k)$	$n \geq 2$	$GL_{n-1}(k)$
$SO_n(k)$	$n \geq 4$	$SO_{n-1}(k)$
$SO_8(k)$		$Spin_7(k)$

which irreducible $k\mathfrak{S}_{r-1}$-modules appear in this multiplicity-free socle of the restriction to $k\mathfrak{S}_{r-1}$ of an arbitrary irreducible $k\mathfrak{S}_r$-module. These results of Kleshchev have recently been extended to the corresponding Hecke algebras of type **A** in [2].

We call a pair (G, H) of connected reductive groups with $H \leq G$ a *multiplicity-free pair* if 1 holds for all Δ_H, ∇_G. The next results give a classification of multiplicity-free pairs. To do this, we first prove a characteristic-free analogue (Theorem 3.5) of a result due to Kimel'fel'd and Vinberg [7] in characteristic 0. In fact, only minor alterations to the original proof are needed in characteristic p. The following characterisation of multiplicity-free pairs is an easy consequence of Theorem 3.5.

Theorem B. *Let $H < G$ be connected reductive algebraic groups. Let B, B_H be Borel subgroups of G, H respectively. Then, (G, H) is a multiplicity-free pair if and only if there is a dense (B, B_H)-double coset in G.*

We now describe the classification of multipicity-free pairs. Theorem B implies that if θ is an isogeny of G, then (G, H) is a multiplicity-free pair if and only if $(\theta(G), \theta(H))$ is a multiplicity-free pair (see Corollary 3.8). So it is sufficient to classify multiplicity-free pairs up to isogenies of G. If (G_1, H_1) and (G_2, H_2) are multiplicity-free pairs then $(G_1 \times G_2, H_1 \times H_2)$ is also a multiplicity-free pair, so that we only need to classify the "indecomposable" multiplicity-free pairs (see (4.3) for a precise definition). Finally, if R is the radical of G, then it is obvious that (G, H) is a multiplicity-free pair if and only if $(G/R, HR/R)$ is a multiplicity-free pair. These reductions show that to classify multiplicity-free pairs, we need only classify the indecomposable multiplicity-free pairs (G, H) with G semisimple and simply connected.

Theorem C. *The indecomposable multiplicity-free pairs (G, H), with G semisimple and simply connected, are precisely the following:*

(i) *The simply connected cover of an entry in table 1.*

(ii) *$G = Sp_{2n}(k)$ and $H = SO_{2n}(k) (p = 2)$.*

(iii) *$G = SL_2(k) \times SL_2(k)$ and H is the diagonal subgroup $\{(g, \theta(g)) \,|\, g \in SL_2(k)\}$, where $\theta : SL_2(k) \to SL_2(k)$ is a Frobenius morphism ($p \neq 0$).*

(iv) *Any pair (G, G) with G simple and simply connected.*

In characteristic 0, Theorem C follows from a result of Krämer [11] which classifies multiplicity-free pairs of compact Lie groups. The possibilities (ii), (iii) in Theorem C only occur in non-zero characteristic.

By definition, if (G, H) is a multiplicity-free pair, then for all Weyl modules Δ_H of H and all coWeyl modules ∇_G of G, $\mathrm{Hom}_H(\Delta_H, \nabla_G) = \mathrm{Ext}^0_H(\Delta_H, \nabla_G)$ is at most 1-dimensional. We next consider higher Ext functors. Call a reductive subgroup $H < G$ a *good filtration subgroup* if $\mathrm{Ext}^i_H(\Delta_H, \nabla_G) = 0$ for all Weyl modules Δ_H of H and all coWeyl modules ∇_G of G, and all $i \geq 1$. This condition is equivalent (eg by [6, II.4.16]) to the property that every coWeyl module ∇_G of G has an H-stable filtration

$$0 = \nabla_0 < \nabla_1 < \cdots < \nabla_n = \nabla_G$$

such that each factor ∇_i / ∇_{i-1} is a coWeyl module for H. Such a filtration is called a *good filtration*, and it is known ([6, II.4.16] again) that the number of factors ∇_i / ∇_{i-1} in the filtration isomorphic to a given coWeyl module ∇_H is equal to $\dim \mathrm{Hom}_H(\Delta_H, \nabla_G)$, where Δ_H is the contravariant dual of ∇_H. Thus, if (G, H) is a multiplicity-free pair such that H is also a good filtration subgroup of G, then in fact every coWeyl module ∇_G of G has a multiplicity-free good filtration as an H-module.

Our final result shows that if (G, H) is a multiplicity-free pair, H is usually a good filtration subgroup. For the case $(G, H) = (SL_n(k), GL_{n-1}(k))$, this result goes back at least to James [4, 26.6], and is in fact a special case of the Donkin-Mathieu restriction theorem [3, 14] which shows that any Levi subgroup of a reductive algebraic group is a good filtration subgroup.

Theorem D. *Let $H < G$ be the simply connected cover of an entry in table 1. Then, H is a good filtration subgroup of G, so that*

$$\mathrm{Ext}^i_H(\Delta_H, \nabla_G) = 0$$

for all Weyl modules Δ_H for H and coWeyl modules ∇_G for G, and all $i \geq 1$. Hence each ∇_G also has a multiplicity-free good filtration as an H-module.

This result extends immediately (by [3, 4.2]) to cover any multiplicity-free pair (G, H) "defined over $\mathbb{Z}$" – that is, no factor in a decomposition of (G, H) into indecomposable multiplicity-free pairs is of type (ii) or (iii) from Theorem C. It is easy to see that these are genuine exceptions: for example if E is the natural $Sp_{2n}(k)$-module and $p = 2$, then $\bigwedge^2 E$ does not have a good filtration as an $SO_{2n}(k)$-module.

2 Proof of Theorem A

2.1. Throughout this note, G will denote a connected reductive algebraic group defined over k. By a G-module, we shall always mean a rational kG-module. Let us fix some notation regarding root systems, Weyl modules etc., following the conventions in Jantzen [6]. Let B be a Borel subgroup of G with unipotent radical U, and let $T < B$ be a maximal torus. Let B^+ be the opposite Borel subgroup to B relative to T, so that $B \cap B^+ = T$. The choice of T determines a root system $\Phi \subset X(T)$, where $X(T)$ is the character group $\mathrm{Hom}(T, k^\times)$. For $\alpha \in \Phi$, let U_α denote the corresponding T-root subgroup of G. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the unique base for Φ such that B is the Borel subgroup generated by *negative* root subgroups. Let $W = N_G(T)/T$ be the Weyl group of G, and fix a positive definite W-invariant symmetric bilinear form $(.,.)$ on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$. For $0 \neq \alpha \in \mathbb{R} \otimes_{\mathbb{Z}} X(T)$, $\alpha^\vee$ denotes $2\alpha/(\alpha, \alpha)$. Let $X_+(T) = \{\lambda \in X(T) \mid (\lambda, \alpha^\vee) \geq 0 \text{ for all } \alpha \in \Pi\}$ be the dominant weights of T (with respect to Π). The choice of Π also fixes a set of simple reflections in W, so that we can talk about the longest element w_0 of W relative to these simple reflections.

Given $\lambda \in X(T)$, let k_λ denote the corresponding 1-dimensional B-module. The coWeyl module $\nabla_G(\lambda)$ is defined to be the induced module $\mathrm{ind}_B^G(k_\lambda)$, and is non-zero precisely when $\lambda \in X_+(T)$ is dominant (see [6, II.2.6]). We will say a G-module is a high weight module of high weight λ if it is generated by a B^+-eigenvector v^+ of weight $\lambda \in X(T)$. Weyl modules are high weight modules, and are 'universal' in the sense that any high weight module is a homomorphic image of some Weyl module [6, II.2.13]. For $\lambda \in X_+(T)$, let $\Delta_G(\lambda)$ denote the corresponding Weyl module of high weight λ, and let $L_G(\lambda)$ denote the simple head of $\Delta_G(\lambda)$ (isomorphic to the simple socle of $\nabla_G(\lambda)$). Finally, the dual $\Delta_G(\lambda)^*$ is isomorphic to $\nabla_G(\lambda^*)$, where $\lambda^* = -w_0\lambda$, by [6, II.2.13].

2.2. The proof of Theorem A depends on the following elementary lemma:

Lemma. *Let (G, H) be a pair of connected reductive algebraic groups with $H < G$. Let B^+, B_H be Borel subgroups of G, H respectively and suppose that the double coset $B_H g B^+$ is dense in G for some $g \in G$. Then,*

$$\dim \operatorname{Hom}_H(\Delta_H, \nabla_G) = \dim \operatorname{Hom}_H(\Delta_G, \nabla_H) \leq 1$$

for all Weyl modules $\Delta_H = \nabla_H^*$ for H and all Weyl modules $\Delta_G = \nabla_G^*$ for G. Hence, the socle and head of $\operatorname{res}^G_H L_G$ are multiplicity-free for every irreducible G-module L_G.

Proof. Let $\Delta_H, \nabla_H, \Delta_G, \nabla_G$ be as in the lemma. The Weyl module Δ_G is generated by some B^+-eigenvector v^+. Since $B_H g B^+$ is dense in G, $\Delta_G = k\text{-span}\{G.v^+\} = k\text{-span}\{B_H g B^+.v^+\} = k\text{-span}\{B_H.gv^+\}$. Hence, Δ_G is generated as a B_H-module by the vector gv^+. Now, by definition ∇_H is an induced module $\operatorname{ind}^H_{B_H} k_\lambda$ for some 1-dimensional B_H-module k_λ. Since any B_H-homomorphism $\Delta_G \to k_\lambda$ is determined by its value on the generator gv^+, and k_λ is 1-dimensional, it is immediate that $\operatorname{Hom}_{B_H}(\Delta_G, k_\lambda)$ is at most 1-dimensional. Applying Frobenius Reciprocity [6, I.3.4] and dualising, we deduce that

$$\dim \operatorname{Hom}_H(\Delta_G, \nabla_H) = \dim \operatorname{Hom}_H(\Delta_H, \nabla_G) \leq 1$$

proving the first part of the lemma.

It remains to show that the socle and head of $\operatorname{res}^G_H L_G$ are multiplicity-free for every irreducible G-module L_G. For the socle, we need to compute $\operatorname{Hom}_H(L_H, L_G)$ for an irreducible H-module L_H. By the universal property of Weyl modules, any homomorphism $L_H \to L_G$ extends to a homomorphism $\Delta_H \to \nabla_G$, where Δ_H is the Weyl module for H with head L_H and ∇_G is the coWeyl module for G with socle L_G. Hence, $\operatorname{Hom}_H(L_H, L_G)$ is also at most 1-dimensional, so that the socle of $\operatorname{res}^G_H(L_G)$ is multiplicity-free. The same argument shows that the head is multiplicity-free, completing the proof. □

2.3. We shall shortly apply this lemma to prove that each entry in table 1 is a multiplicity-free pair. For later use, we shall actually construct a suitable element $g \in G$ explicitly in each case in terms of root subgroups, viewing G as a Chevalley group. Let us briefly recall the construction of Chevalley groups, following Steinberg [18].

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$ and root system $\Phi \subset \mathfrak{h}^*$, and let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a base for Φ as in (2.1). Fix a Chevalley basis $\{X_\alpha, H_i \mid \alpha \in \Phi, 1 \leq i \leq l\}$ for $\mathfrak{g}$ and let $U_\mathbb{Z}$ be the corresponding Kostant $\mathbb{Z}$-form for the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let $\mathfrak{n}^+$ be the subalgebra generated by the X_α with $\alpha \in \Pi$. Fix now some irreducible $\mathfrak{g}$-module $\Delta_\mathbb{C}$ of dimension n, with a fixed high weight vector v^+ annihilated by $\mathfrak{n}^+$. Set $\Delta_\mathbb{Z} = U_\mathbb{Z}.v^+$, an admissible lattice in $\Delta_\mathbb{C}$. Working in a basis of $\Delta_\mathbb{Z}$, we can identify a generator $X_\alpha^i/i!$ of $U_\mathbb{Z}$ ($\alpha \in \Phi, i \geq 0$) with a matrix in $M_n(\mathbb{Z})$, via the representation $\mathfrak{g} \to \mathfrak{gl}(\Delta_\mathbb{C})$. Having done this, the series $\exp(tX_\alpha)$, where t is an indeterminate, has only finitely many non-zero terms, hence gives a well-defined element of $SL_n(\mathbb{Z}[t])$. Set $\Delta_k = \Delta_\mathbb{Z} \otimes_\mathbb{Z} k$; then $x_\alpha(t) = \exp(tX_\alpha)$ defines an automorphism of Δ_k for every $t \in k$. The Chevalley group $G = G_k$ is now defined to be the subgroup of $GL(\Delta_k)$ generated by $\{x_\alpha(t) \mid \alpha \in \Phi, t \in k\}$. It is a semisimple algebraic group over k of the same type as $\mathfrak{g}$, and Δ_k is a Weyl module for G_k.

Now we consider the cases in table 1 in turn.

2.4. For $G = SL_n(k), H = GL_{n-1}(k)$, we make the following choices. Let T be the subgroup of all diagonal matrices, and B be the Borel subgroup of all lower triangular matrices, so that B^+ consists of upper triangular matrices. If $\varepsilon_i : T \to k^\times$ denotes the character $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_i$, we can write $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n\}$ and $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le n-1\}$. Letting $e_1, \dots, e_n$ denote the canonical basis for the natural G-module E, choose H to be the stabiliser of the decomposition $E = \langle e_1, \dots, e_{n-1}\rangle \oplus \langle e_n\rangle$, isomorphic to $GL_{n-1}(k)$. Let $B_H = B \cap H$, a Borel subgroup of H. For $t \in k$, let $x_{\varepsilon_i - \varepsilon_j}(t) \in U_{\varepsilon_i - \varepsilon_j}$ denote the matrix $I + te_{ij}$ (where e_{ij} is the matrix with a 1 in the ij-entry, zeros elsewhere). This is precisely the root group element $x_{\varepsilon_i - \varepsilon_j}(t)$ from the Chevalley construction of (2.3), with the usual choice of Chevalley basis for $\mathfrak{sl}_n(\mathbb{C})$, ie $X_{\varepsilon_i - \varepsilon_j} = e_{ij}$.

Lemma. *With notation as above, the double coset $B_H g B^+$ is dense in G, where*

$$g = x_{\varepsilon_n - \varepsilon_1}(1) x_{\varepsilon_n - \varepsilon_2}(1) \dots x_{\varepsilon_n - \varepsilon_{n-1}}(1).$$

Hence, $(SL_n(k), GL_{n-1}(k))$ is a multiplicity-free pair.

Proof. In terms of the basis $e_1, \dots, e_n$, g is the matrix

$$g = \left(\begin{array}{ccc|c} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right)$$

Since $\dim B_H g B^+ = \dim B_H + \dim B^+ - \dim g^{-1} B_H g \cap B^+ = \dim G - \dim g^{-1} B_H g \cap B^+$, it is sufficient to show $g^{-1} B_H g \cap B^+$ is finite. Suppose $h \in g^{-1} B_H g \cap B^+$; as B^+ consists of upper triangular matrices while $g^{-1} B_H g$ is lower triangular, it follows that $h = \mathrm{diag}(h_1, \dots, h_n)$ is diagonal. Now, $ghg^{-1}.e_i = h_i e_i + (h_i - h_n) e_n$. As $ghg^{-1} \in B_H$, $h_i = h_n$ for $1 \le i \le n-1$. Hence, h lies in the centre $Z(G)$, which is finite. ◻

2.5. For $G = SO_{2n+1}(k), H = SO_{2n}(k)$, fix notation as follows. Let E be the natural $(2n+1)$-dimensional G-module with a G-invariant bilinear form $(.,.)$. Write elements of G as matrices with respect to an ordered basis $e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ for E such that $(e_i, e_j) = 0 (i \ne -j)$, $(e_i, e_{-i}) = 1 (i \ne 0)$ and $(e_0, e_0) = 2$. Let B (resp. B^+) be the lower (resp. upper) triangular matrices in G, a Borel subgroup of G, and T be the diagonal matrices, a maximal torus of G. Let $\varepsilon_i : T \to k^\times$ be the character $\mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \mapsto t_i$. In this notation, the root system Φ can be written as $\{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_k \mid 1 \le i < j \le n, 1 \le k \le n\}$ and $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$.

Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ be the corresponding Lie algebra over $\mathbb{C}$ with natural module E', $\mathfrak{g}$-invariant form $(.,.)'$ and canonical basis $e'_1, \dots, e'_n, e'_0, e'_{-n}, \dots, e'_{-1}$, with properties as in the previous paragraph. Let $E_{i,j} \in \mathfrak{g}$ denote the element such that $E_{i,j}.e'_k = \delta_{jk} e'_i$ for all $-n \le i, j, k \le n$. Then, the elements $X_\alpha (\alpha \in \Phi)$ in table 2 give a Chevalley

Table 2: A Chevalley basis for types B_l, D_l

α	$\varepsilon_i - \varepsilon_j (i < j)$	$\varepsilon_i + \varepsilon_j (i < j)$	ε_i
X_α	$E_{i,j} - E_{-j,-i}$	$E_{j,-i} - E_{i,-j}$	$2E_{i,0} - E_{0,-i}$
$X_{-\alpha}$	$E_{j,i} - E_{-i,-j}$	$E_{-i,j} - E_{-j,i}$	$E_{0,i} - 2E_{-i,0}$

basis for $\mathfrak{g}$ (this is the Chevalley basis used in [5, p38]). The Chevalley construction defines root group elements $x_\alpha(t) \in G$ for $t \in k$ corresponding to this Chevalley basis. We shall only need to use the elements $x_{-\varepsilon_i}(t)$, which act on the natural module E as follows:

$$\begin{aligned} x_{-\varepsilon_i}(t).e_i &= e_i + te_0 - t^2 e_{-i}, \\ x_{-\varepsilon_i}(t).e_0 &= e_0 - 2te_{-i}, \end{aligned}$$

with all other basis elements fixed.

Now, let H be the subgroup $SO_{2n}(k)$ generated by root groups U_α for $\alpha \in \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$. We can describe H geometrically as the connected stabiliser in G of the direct sum decomposition $E = \langle e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\rangle \oplus \langle e_0 \rangle$. Note finally that $B_H = H \cap B$ is a Borel subgroup of H.

Lemma. *With notation as above, the double coset $B_H g B^+$ is dense in G, where*

$$g = x_{-\varepsilon_1}(1) x_{-\varepsilon_2}(1) \dots x_{-\varepsilon_n}(1).$$

Hence, $(SO_{2n+1}(k), SO_{2n}(k))$ is a multiplicity-free pair.

Proof. By the dimension argument of Lemma 2.4, we again just need to show that $g^{-1}B_H g \cap B^+$ is finite. Suppose $h \in g^{-1}B_H g \cap B^+$; as B^+ consists of upper triangular matrices while $g^{-1}B_H g$ is lower triangular, it follows as before that h is diagonal. Let $h = \mathrm{diag}(h_1, \dots, h_n, 1, h_n^{-1}, \dots, h_1^{-1})$. Since $ghg^{-1} \in B_H$, it stabilises $\langle e_1, \dots, e_n, e_{-1}, \dots, e_{-n}\rangle$. The e_0-coefficient of $ghg^{-1}.e_i$ is $h_i - 1$. Hence, $h_i = 1$ for each i, and the intersection is trivial. □

2.6. For $G = SO_{2n}(k), H = SO_{2n-1}(k)$, we shall realise G as the subgroup $SO_{2n}(k)$ constructed in (2.5), acting on the space $E = \langle e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\rangle$. Write elements of G as matrices with respect to this ordered basis for E. Let T, B, B^+ be the diagonal, lower triangular, upper triangular matrices in G respectively, and let $\varepsilon_i, (.,.)$ be the restrictions of those defined in (2.5). We may write $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$ and $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$. Let $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$, with natural module E' and canonical basis $e'_1, \dots, e'_n, e'_{-n}, \dots, e'_{-1}$ corresponding to E, e_i as before. Fix a Chevalley basis for $\mathfrak{g}$ as a subset of the Chevalley basis constructed in (2.5), so that $X_\alpha (\alpha \in \Phi)$ is as in table 2. This gives corresponding parametrisations $x_\alpha(t)$ of the T-root subgroups of G; in particular, for $i < j$, $x_{-\varepsilon_i - \varepsilon_j}(t)$ acts as:

$$\begin{aligned} x_{-\varepsilon_i - \varepsilon_j}(t).e_i &= e_i - te_{-j}, \\ x_{-\varepsilon_i - \varepsilon_j}(t).e_j &= e_j + te_{-i} \end{aligned}$$

with all other basis elements fixed. Let H be the connected stabiliser of $\langle e_n + e_{-n}\rangle$, isomorphic to $SO_{2n-1}(k)$, and note $B_H = B \cap H$ is a Borel subgroup of H. In terms of root subgroups, H is generated by $\{U_{\pm\alpha} \mid \alpha = \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-2} - \varepsilon_{n-1}\}$ together with the elements $\{x_{\varepsilon_{n-1} - \varepsilon_n}(t) x_{\varepsilon_{n-1} + \varepsilon_n}(t), x_{-\varepsilon_{n-1} + \varepsilon_n}(t) x_{-\varepsilon_{n-1} - \varepsilon_n}(t) \mid t \in k\}$.

Lemma. *With notation as above, the double coset $B_H g B^+$ is dense in G, where*

$$g = x_{-\varepsilon_1-\varepsilon_n}(1)x_{-\varepsilon_2-\varepsilon_n}(1)\dots x_{-\varepsilon_{n-1}-\varepsilon_n}(1).$$

Hence, $(SO_{2n}(k), SO_{2n-1}(k))$ and $(SO_8(k), Spin_7(k))$ are multiplicity-free pairs.

Proof. Dimension implies that we just need to show that $g^{-1}B_H g \cap B^+$ is finite. Suppose $h \in g^{-1}B_H g \cap B^+$. The same argument as in (2.5) shows that h is a diagonal matrix, say $h = \operatorname{diag}(h_1, \dots, h_n, h_n^{-1}, \dots, h_1^{-1})$. Now, $ghg^{-1} \in B_H$, so $ghg^{-1}.(e_n + e_{-n}) \in \langle e_n + e_{-n}\rangle$. A direct computation shows that $ghg^{-1}.(e_n + e_{-n}) = h_n e_n + (h_n - h_1^{-1})e_{-1} + \dots + (h_n - h_{n-1}^{-1})e_{-(n-1)} + h_n^{-1}e_{-n}$. Hence, $h_1 = \dots = h_n = \pm 1$ so $g^{-1}B_H g \cap B^+$ is indeed finite.

This proves that $(G, H) = (SO_{2n}(k), SO_{2n-1}(k))$ is a multiplicity-free pair. Now apply a triality graph automorphism (working in $PSO_8(k)$ then taking pre-images since triality is not defined on SO_8 if $p \neq 2$) to deduce that there is also a dense (B_H, B^+)-double coset in G for the pair $(G, H) = (SO_8(k), Spin_7(k))$. Hence, this is also a multiplicity-free pair. □

Lemma 2.4-Lemma 2.6 complete the proof of Theorem A.

3 Proof of Theorem B

To classify multiplicity-free pairs, we first prove an analogue of a result of Kimel'fel'd and Vinberg [7, Theorem 1] in characteristic 0. In this section, we give a proof of this analogue (Theorem 3.5), following the original proof closely, and then deduce Theorem B from it. As always, G denotes a connected reductive algebraic group, with the conventions of (2.1).

3.1. Given an arbitrary closed subgroup $H < G$, we write $X(H) = \operatorname{Hom}(H, k^\times)$ for the character group of H. For any subset $J \subset I = \{1, \dots, l\}$, define the parabolic subgroup $P = P_J$ to be the subgroup generated by B and the T-root subgroups U_{α_j} for $j \in J$. We shall identify $X(P)$ with a subgroup of $X(T)$ via restriction. If H is an arbitrary closed subgroup of G, we define $X^+(H) = \{\lambda \in X(H) \mid \operatorname{ind}_H^G k_\lambda \neq 0\}$. In particular, [6, II.2.6] implies $X_+(T) = X^+(B)$. More generally, if $H = P = P_J$ is parabolic, the following statements are equivalent definitions of $X^+(P)$. Recall here from (2.1) that U denotes the unipotent radical of the negative Borel subgroup B.

(i) $X^+(P) = X(P) \cap X_+(T)$.

(ii) $X^+(P) = \{\lambda \in X_+(T) \mid (\lambda, \alpha_j^\vee) = 0 \text{ for all } j \in J\}$.

(iii) $X^+(P) = \{\lambda \in X_+(T) \mid \Delta_G(\lambda^*)^U \text{ is } P\text{-stable}\}$.

We shall write $k[G]$ (resp. $k(G)$) for the ring of regular (resp. rational) functions on G. We regard $k[G]$ as a G-module in two ways, via the left regular and the right regular representations, where $(g.f)(h) = f(g^{-1}h)$ and $(f.g)(h) = f(hg^{-1})$ for $g, h \in G, f \in k[G]$ respectively. These extend uniquely to define actions of G on $k(G)$. If P, H are any closed subgroups of G, let ${}^P k(G)^H$ be the subalgebra

$${}^P k(G)^H = \{f \in k(G) \mid p.f.h = f, \text{ for all } p \in P, h \in H\}.$$

We shall need Rosenlicht's Theorem [16], which implies that there is a dense (P, H)-double coset in G if and only if ${}^P k(G)^H = k$.

We begin with a basic algebraic lemma.

3.2. **Lemma.** *Let A be a k-algebra that is an integral domain. Let $a, b \in A$ be linearly independent elements. Then, for $n \in \mathbb{Z}^+$, the elements $a^n, a^{n-1}b, \dots, ab^{n-1}, b^n$ are also linearly independent.*

Proof. Let $\sum_{i=0}^s \alpha_i a^i b^{n-i} = 0$ be a dependency with $\alpha_s \neq 0$. Let $\beta_1, \dots, \beta_s$ be the roots of the polynomial $\alpha_0 + \alpha_1 x + \dots + \alpha_s x^s$. Then, $\sum_{i=0}^s \alpha_i a^i b^{n-i} = \alpha_s b^{n-s}(a - \beta_1 b)\dots(a - \beta_s b) = 0$. As A is an integral domain, this implies one of $b, a - \beta_1 b, \dots, a - \beta_s b$ is zero, contradicting the fact that a, b are linearly independent. □

3.3. **Lemma.** *Let H be a closed subgroup of G. Let $\lambda \in X^+(B), \mu \in X(H)$. Write k_μ for the corresponding 1-dimensional H-module. Suppose that* $\dim \mathrm{Hom}_H(\Delta_G(\lambda), k_\mu) \geq 2$. *Then, for all $n \in \mathbb{Z}^+$,* $\dim \mathrm{Hom}_H(\Delta_G(n\lambda), k_{n\mu}) \geq n + 1$.

Proof. Let $\Delta = \Delta_G(\lambda), \Delta_n = \Delta_G(n\lambda)$ and let v^+, w^+ be highest weight vectors in Δ, Δ_n respectively. Let θ_1, θ_2 be linearly independent elements of $\mathrm{Hom}_H(\Delta, k_\mu)$. Let $f_i \in k[G]$ be defined by $f_i(g) = \theta_i(g.v^+)$ for $i = 1, 2$. By the proof of Frobenius reciprocity [6, I.3.4], f_1 and f_2 are linearly independent. Let $\alpha : \Delta_n \to \bigotimes^n \Delta$ be the G-module homomorphism defined by the map $w^+ \mapsto v^+ \otimes \dots \otimes v^+$ and the universal property of Weyl modules. Then, we can define $\phi_i \in \mathrm{Hom}_H(\Delta_n, k_{n\mu})$ for $i = 0, \dots, n$ by composing α with the map $\bigotimes^n \Delta \to k_{n\mu}$ defined by $v_1 \otimes \dots \otimes v_n \mapsto \theta_1(v_1) \otimes \dots \otimes \theta_1(v_i) \otimes \theta_2(v_{i+1}) \otimes \dots \otimes \theta_2(v_n)$. We claim $\phi_0, \dots, \phi_n$ are linearly independent, which will complete the proof. Let $a_0\phi_0 + \dots + a_n\phi_n = 0$ be a dependency. Then, for all $g \in G$,

$$\sum_{i=0}^n a_i \phi_i(g.w^+) = \sum_{i=0}^n a_i \theta_1(g.v^+)^i \theta_2(g.v^+)^{n-i} = 0.$$

So, the element $\sum_{i=0}^n a_i f_1^i f_2^{n-i} \in k[G]$ is zero. But, this implies $a_i = 0$ for each i as the elements $f_1^i f_2^{n-i}$ are linearly independent by the previous lemma. □

3.4. **Remarks.** (I) Let $H < G$ be a connected reductive subgroup. An application of Frobenius reciprocity together with Lemma 3.3 (applied to a Borel subgroup of H) shows that if $\dim \mathrm{Hom}_H(\Delta_H(\mu), \nabla_G(\lambda)) \geq 2$ then $\dim \mathrm{Hom}_H(\Delta_H(n\mu), \nabla_G(n\lambda)) \geq n+1$ for all $n \in \mathbb{Z}^+$. In [11], Krämer uses this to reduce the classification of multiplicity-free pairs (G, H) of compact Lie groups to the case that G is simply connected. We could do this now in our case, but prefer to wait until we can prove the more general Corollary 3.8.

(II) Krämer also introduces the notion of a *multiplicity-bounded* subgroup of a compact connected Lie group. The appropriate analogue in our setting would be a reductive subgroup $H < G$ such that

$$\dim \mathrm{Hom}_H(\Delta_H, \nabla_G) \leq N$$

for all Weyl modules Δ_H for H and coWeyl modules ∇_G for G, where N is some fixed constant independent of Δ_H, ∇_G. By the argument in (I), the concepts of multiplicity-bounded and multiplicity-free subgroups are equivalent.

3.5. **Theorem.** *Let H be an arbitrary closed subgroup of G and $P = P_J$ be the parabolic subgroup of G corresponding to $J \subset I$. The following properties are equivalent.*

(i) $\dim \mathrm{Hom}_H(\Delta_G(\lambda^*), k_\mu) \leq 1$ *for all* $\lambda \in X^+(P), \mu \in X^+(H)$.
(ii) *There is a dense* (P, H)*-double coset in* G.

Proof. (ii)⇒(i). This is just the argument of Lemma 2.2. Recall $\Delta_G(\lambda^*)$ is generated by any vector $0 \neq v \in \Delta_G(\lambda^*)^U$. By (3.1)(iii), v is a P-eigenvector. Hence, if HgP is dense in G, $\Delta_G(\lambda^*)$ is generated as an H-module by the vector gv. This immediately implies that $\mathrm{Hom}_H(\Delta_G(\lambda^*), k_\mu)$ is at most 1-dimensional for any 1-dimensional H-module k_μ.

(i)⇒(ii). We first prove this for G semisimple and simply connected; then, $k[G]$ is a unique factorisation domain by [15]. Suppose there is no dense (P, H)-double coset in G. Then, by Rosenlicht's Theorem, there is some non-constant $f \in {}^P k(G)^H$. Write $f = f_1/f_2$ with $f_1, f_2 \in k[G]$ coprime. Then, for $p \in P, h \in H$, $p.f.h = f$, so $(p.f_1.h)f_2 = f_1(p.f_2.h)$. As $k[G]$ is a unique factorisation domain, this implies that $p.f_i.h = \theta(p, h)f_i$ for each i, where $\theta(p, h) \in k[G]$. Moreover, $\theta(p, h)$ is invertible, and the invertible elements in $k[G]$ are constant. We thus obtain a morphism $\theta : P \times H \to k^\times$, and it is easily checked that this is a character of $P \times H$, so $\theta(p, h) = \lambda(p)\mu(h)$ for characters λ, μ of P, H respectively.

Now, let V_i be the left G-submodule of $k[G]$ generated by f_i. Writing $\dot{w}_0 \in N_G(T)$ for any coset representative of $w_0 \in W$, $\dot{w}_0 f_i$ is a B^+-high weight vector, since f_i is P-stable hence B-stable. So each V_i is a high weight module of high weight $w_0\lambda$. Let $\Delta = \Delta_G(w_0\lambda) = \Delta_G(-\lambda^*)$. By the universal property of Weyl modules, each V_i is a homomorphic image of Δ. By definition of induced module, we can regard each f_i as an element of $\mathrm{ind}_H^G k_\mu$, so that each V_i is a submodule of $\mathrm{ind}_H^G k_\mu$. Thus, we can define two linearly independent homomorphisms $\Delta \to \mathrm{ind}_H^G k_\mu$ by composing $\Delta \to V_i$ with the inclusion $V_i \hookrightarrow \mathrm{ind}_H^G k_\mu$. Now apply Frobenius Reciprocity to show that

$$\dim \mathrm{Hom}_G(\Delta, \mathrm{ind}_H^G k_\mu) = \dim \mathrm{Hom}_H(\Delta, k_\mu) \geq 2.$$

Finally, observe that $-\lambda \in X^+(P)$ by (3.1)(i) and $\mu \in X^+(H)$ by definition. So, this contradicts (i).

Now we treat the general case. Suppose first that G is semisimple and satisfies (i). Let $\tilde{G}$ be the simply connected cover of G. Write $\tilde{H}, \tilde{P}$ for the connected pre-images of H, P respectively in $\tilde{G}$. We just need to show that $(\tilde{G}, \tilde{H})$ also satisfies (i); then, the simply connected result will imply that there is a dense $(\tilde{P}, \tilde{H})$-double coset in $\tilde{G}$, hence that there is a dense (P, H)-double coset in G (this follows as morphisms of algebraic groups are open maps). So, suppose $(\tilde{G}, \tilde{H})$ does not satisfy (i); then there exist $\lambda \in X^+(\tilde{P}), \mu \in X(\tilde{H})$ such that $\dim \mathrm{Hom}_{\tilde{H}}(\Delta_{\tilde{G}}(\lambda^*), k_\mu) \geq 2$. Now, we can choose $n \in \mathbb{Z}^+$ so that $n\lambda, n\mu$ are characters in $X^+(P), X(H)$ respectively. Then, Lemma 3.3 implies $\dim \mathrm{Hom}_H(\Delta_G(n\lambda^*), k_{n\mu}) \geq 2$, a contradiction.

Finally, suppose the radical R of G is non-trivial and that (G, H) satisfies (i). Then clearly $(G/R, HR/R)$ satisfies (i) so the result for semisimple G implies that there is a dense $(P/R, HR/R)$-double coset in G. Taking pre-images, we obtain a dense (P, HR)-double coset in G, hence a dense (P, H)-double coset since $R < P$ is central. □

3.6. **Remarks.** (I) Popov's result [15] shows that if G is semisimple, but not necessarily simply connected, then the divisor class group of G is finite. Using this and a straightforward argument involving divisors, Kimel'fel'd and Vinberg prove (i)⇒(ii) without considering the simply connected case separately.

(II) A subgroup $H < G$ is called *spherical* if there is a dense (H, B)-double coset in G. Spherical subgroups of reductive algebraic groups have been classified in characteristic 0 in [1, 12]. As far as I know, no such classification exists in arbitrary characteristic, even for the special case of reductive spherical subgroups of simple algebraic groups.

(III) Kimel'fel'd and Vinberg also prove that if H is a connected reductive subgroup and $\{\alpha_j \,|\, j \in J\}$ is stable under $-w_0$ (the longest element of W), then (i) is equivalent to

(i)′ $\dim \mathrm{Hom}_H(\Delta_G(\lambda^*), k) = \dim \nabla_G(\lambda)^H \leq 1$ *for all* $\lambda \in X^+(P)$.

This can be proved in arbitrary characteristic providing in addition some conjugate of H is normalised by τ, an anti-automorphism of G (see eg [6, II.1.16]) such that $\tau^2 = 1, {}^\tau t = t$ for $t \in T$ and ${}^\tau U_\alpha = U_{-\alpha}$ for $\alpha \in \Phi$. This extra condition holds for example if H is reductive and of maximal rank in G. Alternatively, in the special case that $P = B$, (i) and (i)′ are equivalent providing H is a closed subgroup such that the field of rational functions $k(G/H)$ is the field of fractions of the regular functions $k[G/H]$ (this includes all reductive subgroups). The proof of this depends on the argument in [7, Theorem 2] (in fact, Kimel'fel'd and Vinberg prove a slightly weaker statement than required here, and consider characteristic 0 only, but the method is easily generalised).

3.7. Now we apply Theorem 3.5 to deduce Theorem B. For the remainder of the section, let $H < G$ be a connected reductive subgroup. Fix a Borel subgroup B_H of H. At this point, we need to talk about root systems, Weyl groups etc. for H as well as for G. Rather than introduce more notation, let us just note that since ind_H^G is exact [6, I.5.12], $X^+(B_H) = \{\lambda \in X(B_H) \,|\, \mathrm{ind}_{B_H}^G k_\lambda \neq 0\}$ also equals $\{\lambda \in X(B_H) \,|\, \mathrm{ind}_{B_H}^H k_\lambda \neq 0\}$. Hence, by (3.1)(i) with $P = B_H$, we can regard $X^+(B_H)$ as an intrinsically defined set of dominant weights for some root system of H, and set $\nabla_H(\lambda) = \mathrm{ind}_{B_H}^H k_\lambda$ for $\lambda \in X^+(B_H)$.

Theorem. *Let $P \geq B$ and $P_H \geq B_H$ be parabolic subgroups of G, H respectively. The following properties are equivalent.*

(i) $\dim \mathrm{Hom}_H(\Delta_G(\lambda^*), \nabla_H(\mu)) \leq 1$ *for all* $\lambda \in X^+(P), \mu \in X^+(P_H)$.

(ii) *There is a dense (P, P_H)-double coset in G.*

Proof. (i)⇒(ii). For $\mu \in X^+(P_H)$, $\nabla_H(\mu) = \mathrm{ind}_{B_H}^H k_\mu = \mathrm{ind}_{P_H}^H k_\mu$. Therefore, we can apply Frobenius reciprocity to (i) to deduce $\dim \mathrm{Hom}_{P_H}(\Delta_G(\lambda^*), k_\mu) \leq 1$ for all $\lambda \in X^+(P), \mu \in X^+(P_H)$. Then, Theorem 3.5 implies there is a dense (P, P_H)-double coset in G.

(ii)⇒(i). Suppose $\dim \mathrm{Hom}_H(\Delta_G(\lambda^*), \nabla_H(\mu)) \geq 2$ for some $\lambda \in X^+(P)$ and $\mu \in X^+(P_H)$. By Frobenius reciprocity again, $\dim \mathrm{Hom}_{P_H}(\Delta_G(\lambda^*), k_\mu) \geq 2$, so there is no dense (P, P_H)-double coset in G by Theorem 3.5. □

Theorem B from the introduction follows immediately from this, putting $P = B$ and $P_H = B_H$. As an immediate corollary, we can show that it is sufficient to consider multiplicity-free pairs up to isogenies of G.

3.8. **Corollary.** *Let θ be an isogeny of G. Then, (G, H) is a multiplicity-free pair if and only if $(\theta(G), \theta(H))$ is a multiplicity-free pair*

Table 3: Reductive subgroups of dimension at least $\frac{1}{2}\dim G$

G	H	G	H
$Sp(E), SO(E)$	N_i	G_2	$A_2, \tilde{A}_2 (p=3)$
$SL(E)$	$Sp(E)$	F_4	$B_4, C_4 (p=2)$
$Sp(E)(p=2)$	$SO(E)$	E_6	F_4
$SO_8(k)$	$Spin_7(k)$	E_7	$A_1 D_6$
$SO_7(k)(p \neq 2), Sp_6(k)(p=2)$	G_2	E_8	$A_1 E_7$

Proof. Let B, B_H be Borel subgroups of G, H respectively. By Theorem B, we need to show that there is a dense (B, B_H)-double coset in G if and only if there is a dense $(\theta(B), \theta(B_H))$-double coset in $\theta(G)$, which is immediate since morphisms of algebraic groups are open maps. □

4 Proof of Theorem C

Theorem B reduces the problem of classifying multiplicity-free pairs to group theory. We shall need to list all reductive subgroups H of simple algebraic groups G satisfying the dimension bound $\dim B + \dim B_H \geq \dim G$ given in Theorem B. Note that we make a distinction between *reductive maximal* subgroups and *maximal reductive* subgroups of G: the former are maximal subgroups of G, whereas the latter may lie in some proper parabolic of G.

4.1. For G classical, we use the notation $G = Cl(E)$ to indicate that G is a connected classical algebraic group with natural module E. When $G = SO(E), Sp(E)$ let N_i denote the connected stabiliser in G of a non-degenerate subspace of E of dimension i with $i \leq \frac{1}{2}\dim E$; and when $(G,p) = (D_n, 2)$ let N_1 denote the connected stabilizer of a nonsingular 1-space. When $p = 3$, we write $\tilde{A}_2$ for the subgroup of G_2 generated by the short root groups relative to some fixed maximal torus.

Lemma. *Let H be a reductive maximal connected subgroup of a simple algebraic group G, and suppose that $\dim H \geq \frac{1}{2}\dim G$. In the case G classical, suppose that $G = Cl(E)$ and that $(G,p) \neq (B_n, 2)$. Then (G,H) are in table 3.*

Proof. For G classical, this is [13, Lemma 5.1]. For G exceptional, it follows from [17] by the argument in [13, Proposition 2.3]. □

4.2. **Lemma.** *The multiplicity-free pairs (G,H) with G simple are precisely those in table 1, up to isogenies of G, together with the trivial case $H = G$ of Theorem C(iv).*

Proof. We exclude the case $(G,H) = (SO_4(k), SO_3(k))$ since here G is not simple. By Theorem B, there is a dense (B, B_H)-double coset in G, so $\dim B + \dim B_H \geq \dim G$. This implies $\dim H \geq \dim G - \operatorname{rank} G - \operatorname{rank} H \geq \dim G - 2 \operatorname{rank} G$. We show that the only pairs (G,H) for which H satisfies this dimension bound are those in table

1 (up to isogenies of G); we already know that all such pairs are multiplicity-free pairs by Theorem A and Corollary 3.8. Note that for each pair (G, H) in table 1, $\dim B + \dim B_H$ exactly equals $\dim G$, so no proper reductive subgroup of H satisfies the dimension bound.

We consider two cases.

(i) Suppose H lies in no parabolic subgroup of G. Then, H lies in some reductive maximal connected subgroup $\bar{H}$ of G. Consider first the possibilities for $\bar{H}$. The bound $\dim \bar{H} \geq \dim G - 2 \operatorname{rank} G$ implies either $(G, \bar{H}) = (SL_2(k), GL_1(k))$ (which is in table 1) or $\dim \bar{H} \geq \frac{1}{2} \dim G$. Hence, $(G, \bar{H})$ are given by Lemma 4.1. Now, one checks that the only possibilities satisfying the stronger dimension bound $\dim \bar{H} \geq \dim G - 2 \operatorname{rank} G$ are those in the conclusion. Hence, $(G, \bar{H})$ is in table 1, and we deduce that $H = \bar{H}$ by dimension.

(ii) Suppose H lies in a maximal parabolic subgroup $P = LQ$ of G, with Levi factor L and unipotent radical Q. Let $\bar{H} \leq L$ be such that $\bar{H}Q/Q = HQ/Q$. Then, H is isogenous to $\bar{H}$ so L also satisfies the dimension bound $\dim L \geq \dim G - 2 \operatorname{rank} G$. Computing the possible dimensions of Levi subgroups, the only possibility is $(G, L) = (SL_n(k), GL_{n-1}(k))$. We deduce that $\bar{H} = L$ by dimension, hence that $H = GL_{n-1}(k)$, which is in table 1. □

4.3. If G is a semisimple algebraic group and $H < G$ is any closed subgroup, we call H a *decomposable subgroup* of G if G, H can be written as commuting products $G = G_1 G_2$, $H = H_1 H_2$ such that, for each i, $H_i \leq G_i$ and $G_i \lhd G$ is a non-trivial semisimple group.

Lemma. *Let $H < G$ be a connected reductive subgroup of a semisimple group G. Suppose H is a decomposable subgroup of G, so that G, H can be written as $G = G_1 G_2, H = H_1 H_2$ as above. Then, (G, H) is a multiplicity-free pair if and only if (G_1, H_1) and (G_2, H_2) are both multiplicity-free pairs.*

Proof. This is immediate from the definition since Weyl modules (resp. coWeyl modules) for G or H are just tensor products of Weyl modules (resp. coWeyl modules) for G_1 and G_2 or H_1 and H_2. □

We define an *indecomposable* multiplicity-free pair to be a multiplicity-free pair (G, H) such that H is an indecomposable subgroup of G. As remarked in the introduction, to classify all multiplicity-free pairs, it is sufficient to classify the indecomposable multiplicity-free pairs (G, H) with G semisimple and simply connected, by Corollary 3.8 and the above Lemma.

The next Lemma is well known.

4.4. **Lemma.** *Let $G = G_1 \dots G_n$ be a semisimple algebraic group written as a commuting product of simple subgroups $G_i \lhd G$, with $n \geq 2$. If H is a maximal connected reductive subgroup of G, then one of the following holds:*

(i) *Some simple factor $1 \neq G_i \lhd G$ is contained in H.*

(ii) *H is diagonally embedded in G and $n = 2$.*

Proof. We may assume G is of adjoint type, so that it is a direct product $G = G_1 \times \cdots \times G_n$ with each G_i simple both as algebraic and abstract groups. We shall write $G^i = \prod_{j \neq i} G_j$. Assume $G_i \not\leq H$ for all i, which immediately implies that H lies in no parabolic subgroup of G. Suppose first that $Z(H) \neq 1$. Take $z = z_1 \dots z_n \in Z(H)$ with $z_i \in G_i$ and $z_j \neq 1$ for some j. As H lies in no parabolic, H is maximal, so $H = C_G(z)^0$. This implies that $H = C_{G_1}(z_1)^0 \dots C_{G_n}(z_n)^0$. Maximality again forces $z_i = 1$ for $i \neq j$, so that $G_i \leq H$ for all $i \neq j$, contradicting our assumption.

So, $Z(H) = 1$ and we can write $H = H_1 \dots H_m$ as a direct product of simple, centreless factors H_i. By maximality, $H = N_G(H_1)^0$. We now show that the projection $\pi_i : H_1 \to G_i$ is a bijection for each i. To see this, notice $H_1 \cap G^i \trianglelefteq H_1$, so equals 1 or H_1, as H_1 is simple as an abstract group. In the latter case, $H_1 \leq G^i$ so $G_i \leq H$, contrary to assumption. So, $H_1 \cap G^i = 1$ and π_i is injective for each i. Next, the normaliser $N_{G/G^i}(H_1G^i/G^i)$ contains HG^i/G^i. But this equals G/G^i by maximality of H, so $H_1G^i/G^i \trianglelefteq G/G^i \cong G_i$. Hence, $H_1G^i/G^i = G/G^i$ and π_i is surjective for each i, as required.

Now let $\theta_i = \pi_i \circ \pi_1^{-1} : G_1 \to G_i$. We have shown that each θ_i is an isomorphism of abstract groups and that $H_1 = \{g\theta_2(g) \dots \theta_n(g) \mid g \in G_1\}$. But then $H = N_G(H_1)^0 = H_1$. Finally, by maximality, we must have that $n = 2$ and (ii) holds. □

4.5. **Lemma.** *Let (G, H) be an indecomposable multiplicity-free pair, such that G is semisimple and simply connected, but not simple. Then, $G = SL_2(k) \times SL_2(k)$ and $H < G$ is a diagonally embedded $SL_2(k)$.*

Proof. First suppose that H is a maximal connected reductive subgroup of G. Then Lemma 4.4 implies that $G = G_1 \times G_2$ is a product of two isomorphic simple factors and H is a diagonally embedded subgroup. Now a routine dimension check shows that the only possibility satisfying the bound $\dim B + \dim B_H \geq \dim G$ is as in the conclusion.

Now suppose for a contradiction that the lemma is false. Then, we can find a counterexample (G, H), such that the lemma holds for all indecomposable multiplicity-free pairs (G_1, H_1) such that either $\dim G_1 < \dim G$ or $\dim G_1 = \dim G$ and $\dim H_1 > \dim H$. By the previous paragraph, H is not a maximal connected reductive subgroup of G, so we may embed $H < K < G$ where K is a connected reductive subgroup of G and H is a maximal connected reductive subgroup of K. Choose Borel subgroups $B_H < B_K < B$ for H, K, G respectively. Obviously, there is a dense (B, B_K)-double coset in G, so (G, K) is a multiplicity-free pair. By Lemma 4.3, we may write G, K as direct products $G = G_1 \times \dots \times G_n$, $K = K_1 \times \dots \times K_n$, such that each pair (G_i, K_i) is an indecompoable multiplicity-free pair. Suppose first that $n = 1$. Then, the minimality hypothesis on (G, H) implies that $G = SL_2(k) \times SL_2(k)$ and K is a diagonally embedded $SL_2(k)$. But for this pair $\dim B + \dim B_K$ is exactly equal to $\dim G$ and B_H is a proper subgroup of B_K. This gives a contradiction, since (G, H) is a multiplicity-free pair.

So $n > 1$. Let Z be the centre of K and, for any subgroup $L \leq K$, denote its image in KZ/Z by L'. The hypothesis on (G, H) implies that the lemma holds for each (G_i, K_i). So each K_i' is simple and in particular H' is an indecomposable subgroup of K', since H is an indecomposable subgroup of G. Now Lemma 4.4 implies that K' is semisimple of length 2 and H' is diagonally embedded in K', as in the first paragraph. The number of factors (G_i, K_i) isomorphic to $(SL_n(k), GL_{n-1}(k))$ is just $\dim Z$, and for these pairs $\dim G_i > \dim K_i + 1$. Hence, $\dim B_H + \dim B \geq \dim G > \dim K + \dim Z$, and this implies that $\dim B_H' + \dim B_K' > \dim K'$. But now the dimension check from the first paragraph gives a contradiction. □

Theorem C follows immediately from Lemma 4.2 and Lemma 4.5.

5 Proof of Theorem D

Let (G, H) be as in Theorem D, and fix notation as in Lemmas (2.4)-(2.6).

5.1. To prove Theorem D, it is sufficient to show that $\nabla_G(\lambda_i)$ has a good filtration as an H-module for each *fundamental* dominant weight $\lambda_i \in X(T)$. This follows by combining the Donkin-Mathieu tensor product theorem (in fact [3, Theorem 4.3.1] is sufficient for our purposes) with the argument of [3, 3.5.4]. We shall prove the equivalent dual statement, that $\Delta_G(\lambda_i)$ has a *Weyl filtration* as an H-module, for each fundamental dominant weight λ_i.

Let us first consider $G = B_l$ or D_l and the fundamental weights λ_l (if $G = B_l$ or D_l) and $\lambda_{l-1}(G = D_l$ only). Spin modules for B_l, D_l are irreducible Weyl modules in all characteristics. If $(G, H) = (D_l, B_{l-1})$ then $\Delta_G(\lambda_{l-1})$ and $\Delta_G(\lambda_l)$ are spin modules, and restrict to the spin module $\Delta_H(\lambda'_{l-1})$ for H. If $(G, H) = (B_l, D_l)$ then $\Delta_G(\lambda_l)$ is a spin module and restricts to a direct sum $\Delta_H(\lambda'_l) \oplus \Delta_H(\lambda'_{l-1})$. Hence $\Delta_G(\lambda_i)$ has a Weyl filtration on restriction to H in each case as required.

It remains to consider the Weyl modules $\Delta_G(\lambda_i)$ for $1 \leq i \leq l$ (if $G = A_l$), $1 \leq i \leq l-1$ (if $G = B_l$) or $1 \leq i \leq l-2$ (if $G = D_l$). Recall from (2.4)-(2.6) that $\mathfrak{g}$ is the corresponding simple Lie algebra over $\mathbb{C}$, with natural module E'. The corresponding irreducible $\mathfrak{g}$-module is just the exterior power $\bigwedge^i E'$ in each case. Now, if $G = SL_n(k)$ or an orthogonal group in characteristic different from 2, $\bigwedge^i E'$ remains irreducible on reduction mod p by [5, p43, Lemma 11], so that $\Delta_G(\lambda_i) \cong \bigwedge^i E$. In each case, an easy argument shows that $\mathrm{res}^G_H \bigwedge^i E \cong \bigwedge^i E_0 \oplus \bigwedge^{i-1} E_0$, where E_0 is the natural module for H; these summands are Weyl modules for H. This completes the proof of Theorem D, unless G is an orthogonal group with $p = 2$.

To include characteristic 2, we now give a short direct argument exploiting the element $g \in G$ in the dense (B_H, B^+)-double coset constructed in Lemmas (2.4)-(2.6). In fact, this argument is valid in all characteristics, and does not depend on the result from [5] used in the previous paragraph. The same argument can also be given for $G = SL_n(k)$.

5.2. **Lemma.** *Let $(G, H) = (SO_{2n}(k), SO_{2n-1}(k))$ or $(SO_{2n+1}(k), SO_{2n}(k))$ with $1 \leq i \leq n-2$ or $1 \leq i \leq n-1$ respectively. Then, $\Delta_G(\lambda_i)$ has a Weyl filtration as an H-module.*

Proof. Let $\Delta_{\mathbb{C}} = \bigwedge^i E'$ be the corresponding irreducible $\mathfrak{g}$-module over $\mathbb{C}$, with notation as in (2.5) or (2.6). Then, $v' = e'_1 \wedge \cdots \wedge e'_i$ is a high weight vector of E', and $\Delta_{\mathbb{Z}} = U_{\mathbb{Z}}.v'$ is an admissible lattice in $\Delta_{\mathbb{C}}$. The Chevalley construction of (2.3) implies that $\Delta = \Delta_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ is the Weyl module $\Delta_G(\lambda_i)$, with high weight vector $v = v' \otimes 1$. Let $T_H = T \cap H$, a maximal torus of H, and B_H^+ be the corresponding opposite Borel subgroup to B_H. Fix a dominance ordering on $X(T_H)$ so that B_H is the Borel subgroup generated by negative T_H-root subgroups.

Now recall the element $g \in G$ from Lemmas (2.5) and (2.6). Since $B_H g B^+$ is dense in G, Δ is generated as a B_H-module by the vector $w = g.v$. There is a canonical way to construct a filtration of Δ using this vector w which we now describe. Write w as a sum $\sum w_\mu$ corresponding to the T_H-weight space decomposition of Δ. Set $\Delta_0 = \{0\}$, and inductively define Δ_i as follows. Pick $\mu_i \in X(T_H)$ maximal with respect to the dominance order on $X(T_H)$ such that $w_{\mu_i} \notin \Delta_{i-1}$. Let Δ_i be the B_H-submodule generated by w_{μ_i} and Δ_{i-1}. This defines an ascending filtration of B_H-modules.

$$\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_m.$$

The construction implies that $w \in \Delta_m$, so that by density, $\Delta_m = \Delta$. The choice of μ_i immediately implies that $w_{\mu_i} + \Delta_{i-1}$ is a B_H^+-eigenvector in Δ/Δ_{i-1} of weight μ_i. Hence, in fact Δ_i/Δ_{i-1} is an H-module, and the filtration is a filtration of H-modules. Each Δ_i/Δ_{i-1} is a high weight module of high weight μ_i, so an image of the Weyl module $\Delta_H(\mu_i)$, and each μ_i must be dominant.

Now work in $\Delta_{\mathbb{Z}}$ to compute the μ_i occuring in the filtration. Since $g \in G$ was constructed as a product of root group elements of the form $x_\alpha(1)$, there is a corresponding element $u \in U_{\mathbb{Z}}$ such that $(u.v') \otimes 1 = g.(v' \otimes 1)$. Let $w' = u.v'$. A short calculation using table 2 shows that w' is the vector

$$(e'_1 - e'_{-n}) \wedge \cdots \wedge (e'_i - e'_{-n})$$

if $G = D_n$, or

$$(e'_1 + e'_0 - e'_{-1}) \wedge (e'_2 + e'_0 - e'_{-2} - 2e'_{-1}) \wedge \cdots \wedge (e'_i + e'_0 - e'_{-i} - 2e'_{-1} - \cdots - 2e'_{-(i-1)})$$

if $G = B_n$. It is straightforward to expand the above expressions and compute the vectors w'_λ occuring in decomposition $w' = \sum w'_\lambda$ corresponding to the weight space decomposition of $\Delta_{\mathbb{Z}}$. Let d' denote the vector e'_{-n} if $G = D_n$ or e'_0 if $G = B_n$. Then, in both cases, the only vectors w'_λ with λ *dominant* are the vectors $e'_1 \wedge \cdots \wedge e'_i$ and $\pm e'_1 \wedge \cdots \wedge e'_{i-1} \wedge d'$ in $\Delta_{\mathbb{Z}}$. Moreover, every vector $w_\lambda \in \Delta$ in the decomposition $w = \sum w_\lambda$ is the image of some vector $w'_\lambda \in \Delta_{\mathbb{Z}}$. This argument shows that the only possibilities for the high weights μ_i occuring in the above filtration are $\varepsilon_1 + \cdots + \varepsilon_i$ and $\varepsilon_1 + \cdots + \varepsilon_{i-1}$.

Now, dimension implies that both of these high weights must indeed occur, and that each factor Δ_i/Δ_{i-1}, which is a high weight module of weight μ_i by construction, must in fact be the Weyl module $\Delta_H(\mu_i)$. Thus, the filtration is a Weyl filtration, completing the proof. □

To complete the proof of Theorem D, it just remains to observe that it also holds for the pair $(G, H) = (SO_8(k), Spin_7(k))$, by applying a triality automorphism to the pair $(SO_8(k), SO_7(k))$ as in Lemma 2.6.

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Department of Mathematics, Imperial College, London, SW7 2BZ.
e-mail: j.brundan@ic.ac.uk