SEMISIMPLIFICATION OF THE CATEGORY OF TILTING MODULES FOR GL_n

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Abstract. We describe the semisimplification of the monoidal category of tilting modules for the algebraic group GL_n in characteristic $p > 0$. In particular, we compute the dimensions of the indecomposable tilting modules modulo p.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and G_n denote the algebraic group $GL_n(\Bbbk)$ for $n \geq 0$. The symmetric tensor category $\mathcal{R}ep(G_n)$ of finite-dimensional rational representations of G_n is a lower finite highest weight category with irreducible, standard, costandard and indecomposable tilting modules $L_n(\lambda)$, $\Delta_n(\lambda)$, $\nabla_n(\lambda)$ and $T_n(\lambda)$ parametrized by their highest weight λ. In the usual coordinates, the dominant weight λ appearing here may be identified with an element of the poset

$$X_n^+ = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \;\middle|\; \lambda_1 \geq \cdots \geq \lambda_n \right\} \tag{1.1}$$

ordered by the usual dominance ordering $\trianglelefteq$. Let $\mathcal{T}ilt(G_n)$ be the full subcategory of $\mathcal{R}ep(G_n)$ consisting of all tilting modules, which is a Karoubian rigid symmetric monoidal category. The defining n-dimensional representation V_n of G_n is an indecomposable tilting module, as are all of its (irreducible) exterior powers and their duals. These modules generate $\mathcal{T}ilt(G_n)$ as a Karoubian monoidal category (i.e., taking tensor products, direct sums and direct summands).

The *semisimplification*

$$\overline{\mathcal{T}ilt(G_n)} := \mathcal{T}ilt(G_n)/\mathcal{N} \tag{1.2}$$

of the category $\mathcal{T}ilt(G_n)$ is its quotient by the tensor ideal $\mathcal{N}$ consisting of all negligible morphisms. This is a semisimple symmetric tensor category with irreducible objects arising from the indecomposable tilting modules whose dimension is non-zero modulo p; see [EO] for further discussion and historical remarks. Of course, if $p = 0$ the category $\mathcal{R}ep(G_n)$ is already semisimple so coincides with the semisimplification $\overline{\mathcal{T}ilt(G_n)}$, and the irreducible objects in $\overline{\mathcal{T}ilt(G_n)}$ are labeled by the set $X_{n,0}^+ := X_n^+$ of all dominant weights. The case $p \geq n$ may also be regarded as classical: in this case, the category $\overline{\mathcal{T}ilt(G_n)}$ is the so-called *Verlinde category*, with irreducible objects arising from the indecomposable tilting modules of highest weight belonging to the set

$$X_{n,p}^+ := \{ \lambda = (\lambda_1, \dots, \lambda_n) \in X_n^+ \mid \lambda_1 - \lambda_n < p - n + 1 \}, \tag{1.3}$$

interpreting $X_{0,p}^+$ as $\{\varnothing\}$. The classical proof of this from [GK, GM] goes as follows. As $X_{n,p}^+$ is the fundamental alcove, the linkage principle implies that $T_n(\lambda) = \Delta_n(\lambda)$ for λ

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in the upper closure $\overline{X}^+_{n,p}$ (defined by replacing $<$ in (1.3) by $\leq$). By the Weyl dimension formula, it follows that $T_n(\lambda)$ is of non-zero dimension modulo p for $\lambda \in X^+_{n,p}$, and its identity morphism is negligible for $\lambda \in \overline{X}^+_{n,p} \setminus X^+_{n,p}$. Then an argument with translation functors gives that the identity morphism of $T_n(\lambda)$ is negligible for any $\lambda \in X^+_n \setminus X^+_{n,p}$, hence, these modules are all of dimension zero modulo p.

In this article, we treat the remaining situations when $0 < p < n$. Note that the case $p = 2$ was worked out already in [EO, §8]. To formulate the main result in general, assume that $n, p > 0$ and let

$$n = n_0 + n_1 p + \cdots + n_r p^r \tag{1.4}$$

be the p-adic decomposition of n, so $0 \leq n_0, \dots, n_{r-1} < p$ and $0 < n_r < p$. We define an embedding

$$\imath : X^+_{n_0} \times X^+_{n_1} \times \cdots \times X^+_{n_r} \hookrightarrow X^+_n \tag{1.5}$$

sending $\underline{\lambda} = \left(\lambda^{(0)}, \dots, \lambda^{(r)}\right)$ to the dominant conjugate of the n-tuple that is the concatenation $\lambda^{(0)} \sqcup \underbrace{\lambda^{(1)} \sqcup \cdots \sqcup \lambda^{(1)}}_{p \text{ copies}} \sqcup \underbrace{\lambda^{(2)} \sqcup \cdots \sqcup \lambda^{(2)}}_{p^2 \text{ copies}} \sqcup \cdots \sqcup \underbrace{\lambda^{(r)} \sqcup \cdots \sqcup \lambda^{(r)}}_{p^r \text{ copies}}$. Let

$$X^+_{n,p} := \imath\left(X^+_{n_0,p} \times \cdots \times X^+_{n_r,p}\right) \subset X^+_n. \tag{1.6}$$

See (5.3)–(5.4) below for a more conceptual description of this set. Also let $\boxtimes$ be the Deligne tensor product of tensor categories (e.g., see [EGNO, §4.6]). The Deligne tensor product of semisimple symmetric tensor categories is again a semisimple symmetric tensor category.

Main Theorem. *For $p > 0$ as above, there is a symmetric monoidal equivalence*

$$\Xi_n : \overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})} \to \overline{\mathcal{T}ilt(G_n)}$$

sending $T_{n_0}(\lambda^{(0)}) \boxtimes \cdots \boxtimes T_{n_r}(\lambda^{(r)})$ for $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r)}) \in X^+_{n_0,p} \times \cdots \times X^+_{n_r,p}$ to $T_n(\imath(\underline{\lambda}))$. In particular, the irreducible objects of $\overline{\mathcal{T}ilt(G_n)}$ are the indecomposable tilting modules with highest weight in $X^+_{n,p}$.

Example. If $p = 5$ and $n = 13 = 3 + 2 \cdot 5$, this implies that $\overline{\mathcal{T}ilt(G_{13})}$ is equivalent to $\overline{\mathcal{T}ilt(G_3)} \boxtimes \overline{\mathcal{T}ilt(G_2)}$. The bijection $\imath : X^+_{3,5} \times X^+_{2,5} \to X^+_{13,5}$ between the labeling sets takes $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}) \in X^+_3 \times X^+_2$ with $\lambda^{(0)}_1 - \lambda^{(0)}_3 < 3$ and $\lambda^{(1)}_1 - \lambda^{(1)}_2 < 4$ to

$$\imath(\underline{\lambda}) = \left(\lambda^{(0)}_1, \lambda^{(0)}_2, \lambda^{(0)}_3, \lambda^{(1)}_1, \lambda^{(1)}_2, \lambda^{(1)}_1, \lambda^{(1)}_2, \lambda^{(1)}_1, \lambda^{(1)}_2, \lambda^{(1)}_1, \lambda^{(1)}_2, \lambda^{(1)}_1, \lambda^{(1)}_2\right)^+ \in X^+_{13}$$

where $+$ denotes dominant conjugate. So $\Xi_{13}(V_3 \boxtimes \Bbbk) \cong V_{13}$, $\Xi_{13}(\Bbbk \boxtimes V_2) \cong \bigwedge^5 V_{13}$ and $\Xi_{13}(V_3 \boxtimes V_2) \cong \bigwedge^6 V_{13} \cong V_{13} \otimes \bigwedge^5 V_{13}$ (isomorphisms in $\overline{\mathcal{T}ilt(G_{13})}$).

Corollary. *If $\lambda \in X^+_n \setminus X^+_{n,p}$ then $\dim T_n(\lambda) \equiv 0 \pmod{p}$. If $\lambda \in X^+_{n,p}$, so that $\lambda = \imath(\underline{\lambda})$ for $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r)}) \in X^+_{n_0,p} \times \cdots \times X^+_{n_r,p}$, then we have that*

$$\dim T_n(\lambda) \equiv \prod_{i=0}^{r} \dim \Delta_{n_i}(\lambda^{(i)}) \pmod{p}.$$

The right hand side here may be computed explicitly using the Weyl dimension formula.

Proof. For each $i = 0, \dots, r$, we have that $p > n_i$, so by the classical description of Verlinde categories we have that $\dim T_{n_i}(\lambda^{(i)}) \equiv \dim \Delta_{n_i}(\lambda^{(i)}) \pmod{p}$ for $\lambda^{(i)} \in X^+_{n_i}$. Now the corollary follows from the theorem since symmetric monoidal functors are trace-preserving, hence, they also respect categorical dimensions. □

The Main Theorem gives rise to a categorification of Lucas' theorem in the following sense. If $k = k_0 + k_1 p + \cdots + k_r p^r$ for $0 \leq k_0, \ldots, k_r < p$, then $\bigwedge^k V_n \in \overline{\mathcal{T}ilt(G_n)}$ is the image of the irreducible object $\bigwedge^{k_0} V_{n_0} \boxtimes \cdots \boxtimes \bigwedge^{k_r} V_{n_r} \in \overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})}$ under the equivalence Ξ_n from the theorem. We deduce on taking categorical dimensions that

$$\binom{n}{k} \equiv \prod_{i=0}^{r} \binom{n_i}{k_i} \pmod p, \tag{1.7}$$

which is exactly the *classical Lucas theorem*.

An essential step in the proof is provided by a theorem of Donkin from [D1], which gives a version of skew Howe duality for the general linear group. In fact, we rephrase Donkin's result in terms of what we call the *Schur category*; see Theorem 4.14 for the statement. The Schur category is a strict monoidal category closely related to the classical Schur algebra; see Definition 4.2. It also has an explicit diagrammatic realization in terms of webs, which is due to Cautis, Kamnitzer and Morrison [CKM]. Since we are working in positive characteristic, we have included a self-contained treatment establishing the connection between the Schur category and webs via an approach which is independent of [CKM]; see Theorem 4.10.

The Main Theorem reduces the study of $\overline{\mathcal{T}ilt(G_n)}$ for all $p \geq 0$ to the classical cases in which $p = 0$ or $p > n$. In these classical cases, it can be helpful to think about the combinatorial structure of $\overline{\mathcal{T}ilt(G_n)}$ from the perspective of categorification. Let $\mathfrak{s}$ be the affine Kac-Moody algebra $\mathfrak{sl}_\infty$ if $p = 0$ or $\widehat{\mathfrak{sl}}_p$ if $p > n$, with fundamental weights Λ_i and simple coroots h_i for $i \in \mathbb{Z}/p\mathbb{Z}$. There is a well-known categorical action making $\mathcal{R}ep(G_n)$ into a 2-representation of the Kac-Moody 2-category $\mathfrak{U}(\mathfrak{s})$. (The quickest way to construct this is to apply [BSW, Theorem 4.11], starting from the action of the degenerate Heisenberg category of central charge zero under which $\uparrow$ acts by tensoring with V_n and $\downarrow$ acts by tensoring with V_n^*, as is discussed in the introduction of [BSW].) This categorical action restricts to give an action of $\mathfrak{U}(\mathfrak{s})$ on $\mathcal{T}ilt(G_n)$ such that

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{T}ilt(G_n)) \cong \bigwedge\nolimits^n \mathrm{Nat}_p \tag{1.8}$$

as an $\mathfrak{s}$-module, where Nat_p is a natural level zero representation of $\mathfrak{s}$ with basis $(m_i)_{i \in \mathbb{Z}}$ such that m_i is of weight $\Lambda_{i-1} - \Lambda_i$; see the discussion in the introduction of [B], or [RW, Proposition 6.5]. In particular, $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{T}ilt(G_n))$ is generated as an $\mathfrak{s}$-module by the class $[\Bbbk]$ of the trivial module, which corresponds under (1.8) to the vector $m_0 \wedge m_{-1} \wedge \cdots \wedge m_{1-n} \in \bigwedge^n \mathrm{Nat}_p$ of weight $\Lambda_{-n} - \Lambda_0$. The ideal $\mathcal{N}$ of negligible morphisms defines a sub-2-representation, hence, the quotient $\overline{\mathcal{T}ilt(G_n)}$ is a 2-representation as well. Its complexified Grothendieck ring satisfies

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\overline{\mathcal{T}ilt(G_n)}) \cong V(\Lambda_{-n} - \Lambda_0), \tag{1.9}$$

i.e., it is the level zero extremal weight module parametrized by the minuscule weight $\Lambda_{-n} - \Lambda_0$ in the sense of [K]. This follows because, as an $\mathfrak{s}$-module, $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\overline{\mathcal{T}ilt(G_n)})$ is generated by a vector of weight $\Lambda_{-n} - \Lambda_0$, and it is minuscule as all of its weights λ satisfy $\langle h_i, \lambda \rangle \in \{0, 1, -1\}$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. The latter assertion follows from the semisimplicity of the category $\overline{\mathcal{T}ilt(G_n)}$ by invoking some of the general structure theory of Kac-Moody 2-representations. In more detail, semisimplicity implies that the representation-theoretic Kashiwara operators ε_i, ϕ_i as defined e.g. in [BSW, §5.1] satisfy $\varepsilon_i(L), \phi_i(L) \leq 1$ for all irreducible objects $L \in \overline{\mathcal{T}ilt(G_n)}$ and all $i \in \mathbb{Z}/p\mathbb{Z}$. Since the weight λ of the class of L in $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\overline{\mathcal{T}ilt(G_n)})$ satisfies $\langle h_i, \lambda \rangle = \phi_i(L) - \varepsilon_i(L)$ by [BSW, Lemma 5.2], this implies that $\langle h_i, \lambda \rangle \in \{0, 1, -1\}$ for all i.

We remark finally that there is also a generalization of our Main Theorem to the quantum general linear group $G_{n,q}$ for any $q \in \Bbbk^\times$ such that q^2 is a primitive ℓth root of unity. It is related to the *quantum Lucas theorem*. The proof in the quantum case is quite similar, using Donkin's skew Howe duality established in [D2] formulated in terms of the *q-Schur category*, which again can be viewed diagrammatically in terms of the webs of [CKM]. This will be developed in a subsequent paper.

Conventions. All categories will be $\mathbb{k}$-linear with finite-dimensional Hom-spaces, and all functors will be $\mathbb{k}$-linear. A category is *Karoubian* if it is additive and idempotent complete. Functors between Karoubian categories are automatically additive due to the assumption that they are $\mathbb{k}$-linear,

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2. Background about semisimplification

In this section, we give a self-contained treatment of some basic facts about semisimplification which will be needed later. The results here are all well known and first appeared in [BW] (see also [D, §6] and [AK]). We work in the setting of symmetric monoidal categories for simplicity, but the arguments are quite general. For further discussion of the extension to pivotal categories, see [EO, §2.3].

Following our general conventions, all monoidal categories will be $\mathbb{k}$-linear, meaning in particular that the tensor product functor $-\otimes-$ is bilinear, with finite-dimensional Hom-spaces. A *tensor category* means a monoidal category which is rigid and Abelian, with all objects having finite length, and satisfying $\operatorname{End}(\mathbb{1}) = \mathbb{k}$. Note that in such a category the functor $-\otimes-$ is biexact. See [EGNO, Ch. 4] for a detailed treatment.

Let $\mathcal{D}$ be a rigid symmetric monoidal category with $\operatorname{End}_{\mathcal{D}}(\mathbb{1}) = \mathbb{k}$. By the *trace* $\operatorname{Tr}(f)$ of a morphism $f : X \to X$, we mean the scalar in $\mathbb{k}$ defined by the composition

$$\mathbb{1} \stackrel{\operatorname{coev}_X}{\longrightarrow} X \otimes X^* \stackrel{f\otimes \operatorname{id}_{X^*}}{\longrightarrow} X \otimes X^* \stackrel{s_{X,X^*}}{\longrightarrow} X^* \otimes X \stackrel{\operatorname{ev}_X}{\longrightarrow} \mathbb{1},$$

where coev_X and ev_X are the evaluation and coevaluation morphisms for the dual X^* of X, and $s_{X,Y} : X \otimes Y \stackrel{\sim}{\to} Y \otimes X$ is the symmetric braiding. Then the *categorical dimension* $\operatorname{Dim} X$ means $\operatorname{Tr}(\operatorname{id}_X)$. Note that symmetric monoidal functors between categories of this sort preserve trace, hence also categorical dimensions. The category $\mathcal{D} = \mathcal{T}ilt(G_n)$ considered later in the paper admits a symmetric monoidal functor to vector spaces ("fiber functor"), so for $V \in \mathcal{T}ilt(G_n)$ the categorical dimension $\operatorname{Dim} V$ coincides with the image in $\mathbb{k}$ of the usual dimension $\dim V$ of the underlying vector space.

A category $\mathcal{A}$ is *semisimple* if it is Abelian and every object is isomorphic to a finite direct sum of irreducible objects. In a semisimple category, every short exact sequence splits. The following lemma is taken from [M, Section 2.1].

Lemma 2.1. *Let $\mathcal{A}$ be a $\mathbb{k}$-linear category with finite-dimensional* Hom*-spaces. Then $\mathcal{A}$ is semisimple if and only if it is Karoubian, there exists a family $(L_i)_{i\in I}$ of objects such that $\dim \operatorname{Hom}_{\mathcal{A}}(L_i, L_j) = \delta_{i,j}$ for all $i, j \in I$, and moreover any object of $\mathcal{A}$ is isomorphic to a finite direct sum of objects L_i $(i \in I)$.*

Remark 2.2. The last condition in Lemma 2.1 may be replaced by the following: for all $U, V \in \mathcal{A}$ the map

$$\bigoplus_{i\in I} \operatorname{Hom}_{\mathcal{A}}(U, L_i) \otimes_{\mathbb{k}} \operatorname{Hom}_{\mathcal{A}}(L_i, V) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(U, V)$$

given by composition is an isomorphism.

Definition 2.3. Let $\mathcal{D}$ be a Karoubian rigid symmetric monoidal category satisfying $\operatorname{End}_{\mathcal{D}}(\mathbb{1}) = \mathbb{k}$. For any $X, Y \in \mathcal{D}$, we let

$$\mathcal{N}(X, Y) := \big\{ f : X \to Y \bigm| \operatorname{Tr}(g \circ f) = 0 \text{ for all } g : Y \to X \big\}$$

and denote by $\mathcal{N}$ the corresponding collection of $\mathcal{N}(X,Y)$ over all $X,Y \in \mathcal{D}$. Then $\mathcal{N}$ is a tensor ideal (see e.g. [EO, Lemma 2.3]), called the *tensor ideal of negligible morphisms* in $\mathcal{D}$. We define the *semisimplification* of $\mathcal{D}$ to be the quotient category

$$\overline{\mathcal{D}} = \mathcal{D}/\mathcal{N},$$

letting $Q : \mathcal{D} \to \overline{\mathcal{D}}$ be the canonical quotient functor. In particular, this means that the object set of $\overline{\mathcal{D}}$ is the same as for $\mathcal{D}$, i.e., $QX = X$ for all $X \in \mathcal{D}$, although of course non-isomorphic objects of $\mathcal{D}$ may be isomorphic in $\overline{\mathcal{D}}$.

The category $\overline{\mathcal{D}}$ in Definition 2.3 is again a Karoubian rigid symmetric monoidal category with $\mathrm{End}_{\overline{\mathcal{D}}}(\mathbb{1}) = \Bbbk$ (see e.g. [D, §6]). Also the quotient functor Q is a full symmetric monoidal functor.

Lemma 2.4. *Let $\mathcal{D}$ be as in Definition 2.3, and assume moreover that all nilpotent endomorphisms in $\mathcal{D}$ have trace zero. Let $X \in \mathcal{D}$ be an indecomposable object with endomorphism algebra $E := \mathrm{End}_{\mathcal{D}}(X)$, and $J := J(E)$ be the Jacobson radical.*

1. *If $\mathrm{Dim}\, X \neq 0$ then $\mathcal{N}(X,X) = J$, hence, $\dim \mathrm{End}_{\overline{\mathcal{D}}}(X) = 1$.*
2. *If $\mathrm{Dim}\, X = 0$ then $\mathcal{N}(X,X) = E$, hence, $\dim \mathrm{End}_{\overline{\mathcal{D}}}(X) = 0$.*
3. *Given another indecomposable object $Y \not\cong X$, all morphisms $X \to Y$ are negligible, hence, $\dim \mathrm{Hom}_{\overline{\mathcal{D}}}(X,Y) = 0$.*

Proof. Since E is finite-dimensional and local over an algebraically closed field, its Jacobson radical is of codimension one. The assumption on $\mathcal{D}$ implies that all elements of J are of trace zero. Since J is an ideal, we deduce that $J \le \mathcal{N}(X,X) \le E$.

(1) As $\mathrm{Dim}\, X \neq 0$, the identity endomorphism 1_E of X is not negligible. Hence, $\mathcal{N}(X,X) \neq E$, so we must have that $\mathcal{N}(X,X) = J$.

(2) We must show that $\mathrm{Tr}(f) = 0$ for all $f \in E$. To see this, write f as $\lambda 1_E + h$ for $\lambda \in \Bbbk$ and $h \in J$. Then $\mathrm{Tr}(f) = \mathrm{Tr}(\lambda 1_E + h) = \lambda\, \mathrm{Dim}\, X + \mathrm{Tr}(h) = 0$.

(3) We must show that $\mathrm{Tr}(g \circ f) = 0$ for any morphisms $f : X \to Y$ and $g : Y \to X$. Note that $g \circ f$ is not an isomorphism, since otherwise f would be a split embedding of X into Y with left inverse $(g \circ f)^{-1} \circ g$, contradicting the assumption that X and Y are indecomposable with $X \not\cong Y$. Hence, $g \circ f \in J$, which we have already observed is contained in $\mathcal{N}(X,X)$. □

Theorem 2.5. *For $\mathcal{D}$ as in Definition 2.3, the following conditions are equivalent:*

1. *$\overline{\mathcal{D}}$ is a semisimple symmetric tensor category;*
2. *there exists a symmetric monoidal functor from $\mathcal{D}$ to a symmetric tensor category;*
3. *all nilpotent endomorphisms in $\mathcal{D}$ have trace zero.*

When these conditions hold, the irreducible objects in $\overline{\mathcal{D}}$ are the indecomposable objects of $\mathcal{D}$ of non-zero dimension, two such objects being isomorphic in $\overline{\mathcal{D}}$ if and only if they are isomorphic in $\mathcal{D}$.

Proof. The implication (1) ⇒ (2) follows because $Q : \mathcal{D} \to \overline{\mathcal{D}}$ is such a functor. The implication (2) ⇒ (3) follows from the fact that in a tensor category, any nilpotent endomorphism has trace zero (see [D, §6]). For the remainder of the proof, we assume (3) and must prove (1) together with the final assertion.

The category $\mathcal{D}$ is Krull-Schmidt. In particular, any object is a finite direct sum of indecomposable objects. This follows from the finite-dimensionality of the endomorphism algebras $\mathrm{End}_{\mathcal{D}}(X)$ for all $X \in \mathcal{D}$. In view of Lemma 2.4(2), indecomposable objects of $\mathcal{D}$ with categorical dimension zero become zero objects in $\overline{\mathcal{D}}$. Thus, if we let $(L_i)_{i \in I}$ be a system of representatives for the isomorphism classes of indecomposable objects of non-zero categorical dimension in $\mathcal{D}$, we deduce that every object of $\overline{\mathcal{D}}$ is isomorphic to a finite direct sum of $L_i\,(i \in I)$. The other parts of Lemma 2.4 check the remaining hypothesis $\dim \mathrm{Hom}_{\overline{\mathcal{D}}}(L_i, L_j) = \delta_{i,j}$ of Lemma 2.1, thereby showing that $\overline{\mathcal{D}}$ is semisimple. The final assertion follows by Lemma 2.4 again. □

Finally, we record the following, which makes the universal property of the semisimplification $\overline{\mathcal{D}}$ explicit.

Lemma 2.6. *Suppose that $\mathcal{D}$ satisfies the conditions of Theorem 2.5. Let $F : \mathcal{D} \to \mathcal{A}$ be a full symmetric monoidal functor to a semisimple symmetric tensor category $\mathcal{A}$. Then there is a unique fully faithful symmetric monoidal functor $U : \overline{\mathcal{D}} \longrightarrow \mathcal{A}$ such that $F = U \circ Q$.*

Proof. Let $\mathcal{I}$ be the kernel of F, that is, the collection of all morphisms f in $\mathcal{D}$ which are annihilated by the functor $F : \mathcal{D} \to \mathcal{A}$. Given $f : X \to Y$ in $\mathcal{I}$, we have that $\mathrm{Tr}(g \circ f) = \mathrm{Tr}(F(g) \circ F(f)) = \mathrm{Tr}(0) = 0$ for all $g : Y \to X$. Hence, $\mathcal{I} \subseteq \mathcal{N}$. As the functor F is full, the image under F of any $f \in \mathcal{N}$ is negligible in $\mathcal{A}$ as well. On the other hand, $\mathcal{A}$ is semisimple, so it has no non-zero negligible morphisms (see [D, §6]). Hence, $\mathcal{I} = \mathcal{N}$.

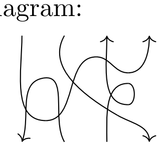
Now to prove the lemma, note that the objects of $\overline{\mathcal{D}}$ are the same as the objects of $\mathcal{D}$, so we must take $UX := FX$ for $X \in \mathcal{D}$. Then on a morphism $\bar{f} \in \mathrm{Hom}_{\overline{\mathcal{D}}}(X, Y)$, we must take $U(\bar{f}) := F(f)$ where $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ is any lift chosen so that $Q(f) = \bar{f}$. By the previous paragraph, this is well-defined and faithful. □

3. Construction of the equivalence

Given a parameter $t \in \Bbbk$, the *oriented Brauer category* $\mathcal{OB}(t)$ is the free rigid symmetric monoidal category generated by an object of categorical dimension t. It can be realized explicitly using the usual string calculus for strict monoidal categories, as follows. The objects of $\mathcal{OB}(t)$ are words in the symbols $\uparrow$ (the generating object) and $\downarrow$ (its dual). For two such words $X = X_1 \cdots X_r$ and $Y = Y_1 \cdots Y_s$, an $X \times Y$ *oriented Brauer diagram* is a diagrammatic representation of a bijection

$$\{i \mid X_i = \uparrow\} \sqcup \{j \mid Y_j = \downarrow\} \xrightarrow{\sim} \{i \mid X_i = \downarrow\} \sqcup \{j \mid Y_j = \uparrow\}$$

obtained by placing vertices labeled in order from left to right according to the letters of the word X (resp., Y) on the top (resp., bottom) boundary, then connecting these vertices with strings as prescribed by the given bijection. For example, the following is a $\downarrow\downarrow\uparrow\uparrow \times \downarrow\uparrow\uparrow\downarrow$ oriented Brauer diagram:

 .

Two $X \times Y$ oriented Brauer diagrams are *equivalent* if they represent the same bijection. The morphism space $\mathrm{Hom}_{\mathcal{OB}(t)}(Y, X)$ is the vector space with basis given by the equivalence classes $[f]$ of $X \times Y$ oriented Brauer diagrams. The tensor product $[f] \otimes [g]$ of two morphisms is the equivalence class defined by the horizontal concatenation of the diagrams f and g. The composition $[f] \circ [g]$ is obtained by vertically stacking the diagram f on top of g then removing closed bubbles in the interior of the diagram, multiplying by t each time a bubble is removed. Alternatively, the category $\mathcal{OB}(t)$ can be defined rather concisely by generators and relations; see [BCNR].

Let $\mathrm{Kar}(\mathcal{OB}(t))$ be the Karoubi envelope of $\mathcal{OB}(t)$, that is, the idempotent completion of its additive envelope. When $\Bbbk$ is of characteristic zero, this category is better known as the *Deligne category* $\underline{\mathcal{R}ep}\,(GL_t)$, but since we are most interested in the positive characteristic case we will avoid this terminology[1]. The category $\mathrm{Kar}(\mathcal{OB}(t))$ is

[1] The appropriate analog of the Deligne category in positive characteristic is bigger than $\mathrm{Kar}(\mathcal{OB}(t))$.

relevant to the problem in hand since, taking t to be the image of $n \in \mathbb{N}$ in the field $\Bbbk$, there is a symmetric monoidal functor

$$\Psi_n : \mathrm{Kar}(\mathcal{OB}(t)) \to \mathcal{T}ilt(G_n) \tag{3.1}$$

sending $\uparrow$ to the natural G_n-module V_n and $\downarrow$ to the dual module V_n^*. By a version of Schur-Weyl duality, this functor is *full*, and it is *dense* if either $p = 0$ or $p > n$; e.g., see [B].

Remark 3.1. When $p = 0$ or $p > n$ (and t is the image of n in $\Bbbk$ still), the functor Ψ_n induces an equivalence of symmetric monoidal categories between $\mathrm{Kar}(\mathcal{OB}(t)/\mathcal{I}_n)$ and $\mathcal{T}ilt(G_n)$, where $\mathcal{I}_n$ is the tensor ideal of $\mathcal{OB}(t)$ generated by the endomorphism of $\uparrow^{\otimes(n+1)}$ associated to the quasi-idempotent $\sum_{g \in S_{n+1}} (-1)^{\ell(g)} g$ in the group algebra $\Bbbk S_{n+1}$ of the symmetric group. This is explained in detail in [B]. This article also constructs a categorical action of the Kac-Moody 2-category $\mathfrak{U}(\mathfrak{s})$ on $\mathrm{Kar}(\mathcal{OB}(t))$ in the same spirit as (1.8)–(1.9), showing that

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{Kar}(\mathcal{OB}(t))) \cong V(-\Lambda_0) \otimes V(\Lambda_{-n}) \tag{3.2}$$

as an $\mathfrak{s}$-module, i.e., it is the tensor product of the integrable lowest weight module of lowest weight $-\Lambda_0$ and the integrable highest weight module of highest weight Λ_{-n}.

Lemma 3.2. *Assume that $t \in \Bbbk$ is the image of $n \in \mathbb{N}$. Then the semisimplifications $\overline{\mathrm{Kar}(\mathcal{OB}(t))}$ and $\overline{\mathcal{T}ilt(G_n)}$ are semisimple symmetric tensor categories. Moreover, if $p = 0$ or $p > n$, the functor Ψ_n induces an equivalence of symmetric monoidal categories*

$$\overline{\Psi}_n : \overline{\mathrm{Kar}(\mathcal{OB}(t))} \to \overline{\mathcal{T}ilt(G_n)}.$$

Proof. Since $\mathcal{T}ilt(G_n)$ embeds into the tensor category $\mathcal{R}ep(G_n)$, we get that $\overline{\mathcal{T}ilt(G_n)}$ is a semisimple symmetric tensor category by Theorem 2.5. Similarly, we get that $\overline{\mathrm{Kar}(\mathcal{OB}(t))}$ is a semisimple symmetric tensor category by considering the composition of the symmetric monoidal functor (3.1) with the inclusion of $\mathcal{T}ilt(G_n)$ into $\mathcal{R}ep(G_n)$. If $p = 0$ or $p > n$ then Ψ_n is full and dense, hence, so too is

$$\widetilde{\Psi}_n := Q \circ \Psi_n : \mathrm{Kar}(\mathcal{OB}(t)) \to \overline{\mathcal{T}ilt(G_n)}.$$

Applying Lemma 2.6, this descends to give the symmetric monoidal equivalence $\overline{\Psi}_n$. □

When $0 < p \leq n$, the functor Ψ_n is no longer dense. To rectify this, we need to work more generally with the *colored oriented Brauer category* $\mathcal{OB}(t_0, \dots, t_r)$, that is, the free rigid symmetric monoidal category generated by $(r+1)$ objects $\uparrow_0, \dots, \uparrow_r$ of dimensions $t_0, \dots, t_r \in \Bbbk$, respectively. The definition of this is similar to $\mathcal{OB}(t)$, except that now strings are labeled by an additional color from the set $\{0, \dots, r\}$. Thus, $\mathcal{OB}(t_0, \dots, t_r)$ has generating objects $\{\uparrow_i, \downarrow_i \mid i = 0, \dots, r\}$, and morphisms are $\Bbbk$-linear combinations of equivalence classes of *colored oriented Brauer diagrams*. Horizontal and vertical composition are as before; in the latter case, one multiplies by the parameter t_i each time a closed bubble of color i is removed.

Lemma 3.3. *Suppose that $t_0, \dots, t_r \in \Bbbk$ are the images of $n_0, \dots, n_r \in \mathbb{N}$. Then the semisimplification $\overline{\mathrm{Kar}(\mathcal{OB}(t_0, \dots, t_r))}$ is a semisimple symmetric tensor category. Moreover, assuming either $p = 0$ or $p > \max(n_0, \dots, n_r)$, there is an equivalence of symmetric monoidal categories*

$$\overline{\Psi}_{n_0,\dots,n_r} : \overline{\mathrm{Kar}(\mathcal{OB}(t_0, \dots, t_r))} \to \overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})}.$$

sending $\uparrow_i$ to V_{n_i}, the natural G_{n_i}-module, and $\downarrow_i$ to $V_{n_i}^*$.

Proof. By universal properties, there is a symmetric monoidal functor

$$\widetilde{\Psi}_{n_0,\dots,n_r} : \mathrm{Kar}(\mathcal{OB}(t_0, \dots, t_r)) \to \overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})}$$

sending $\uparrow_i$ to V_{n_i} and $\downarrow_i$ to $V_{n_i}^*$. If $p = 0$ or $p > \max(n_0, \dots, n_r)$, the symmetric monoidal functors $\Psi_{n_i} : \mathrm{Kar}(\mathcal{OB}(t_i)) \to \mathcal{T}ilt(G_{n_i})$ defined as in (3.1) are all full and dense, hence, $\widetilde{\Psi}_{n_0,\dots,n_r}$ is full and dense too. Since $\overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})}$ is a semisimple symmetric tensor category, Theorem 2.5 implies that $\overline{\mathrm{Kar}(\mathcal{OB}(t_0,\dots,t_r))}$ is a semisimple symmetric tensor category. Finally, Lemma 2.6 gives that $\widetilde{\Psi}_{n_0,\dots,n_r}$ descends to the desired equivalence $\overline{\Psi}_{n_0,\dots,n_r}$. $\square$

Now we can explain the strategy for the construction of the equivalence Ξ_n in the Main Theorem. Assume that $p > 0$ and fix a p-adic decomposition of n as in (1.4). Let $t_i \in \Bbbk$ be the image of n_i. By a special case of (1.7), we have that

$$\dim \textstyle\bigwedge^{p^i} V_n = \binom{n}{p^i} \equiv n_i \pmod{p}.$$

Hence, there is a symmetric monoidal functor

$$\Phi_n : \mathrm{Kar}(\mathcal{OB}(t_0,\dots,t_r)) \to \mathcal{T}ilt(G_n) \tag{3.3}$$

sending $\uparrow_i$ to $\bigwedge^{p^i} V_n$ and $\downarrow_i$ to $\bigwedge^{p^i} V_n^*$.

Lemma 3.4. *In the setup of (1.4), the category $\overline{\mathcal{T}ilt(G_n)}$ is generated as a Karoubian monoidal category by the exterior powers $\bigwedge^{p^i} V_n$ of the natural G_n-module V_n and their duals for $i = 0,\dots,r$.*

Proof. By highest weight considerations, the Karoubian monoidal category $\mathcal{T}ilt(G_n)$ is generated by the exterior powers $\bigwedge^k V_n$ and their duals for $k = 1,\dots,n$. By Lucas' theorem (1.7), $\dim \bigwedge^k V_n \equiv 0 \pmod{p}$, hence, $\bigwedge^k V_n$ is zero in $\overline{\mathcal{T}ilt(G_n)}$, unless $k = k_0 + k_1 p + \cdots + k_r p^r$ for $0 \le k_0 \le n_0, \dots, 0 \le k_r \le n_r$. Therefore, $\overline{\mathcal{T}ilt(G_n)}$ is generated by the exterior powers $\bigwedge^k V_n$ and their duals for k of this special form. To complete the proof, we show for any such k that $\bigwedge^k V_n$ is a summand of the tilting module

$$T := (V_n)^{\otimes k_0} \otimes (\textstyle\bigwedge^p V_n)^{\otimes k_1} \otimes \cdots \otimes \left(\bigwedge^{p^r} V_n\right)^{\otimes k_r}.$$

For each i, we have that $k_i < p$, hence, $W_i := \bigwedge^{k_i p^i} V_n$ is the summand of $\left(\bigwedge^{p^i} V_n\right)^{\otimes k_i}$ defined by the idempotent $e_i := \frac{1}{k_i!}\sum_{g \in S_{k_i}} (-1)^{\ell(g)} g \in \Bbbk S_{k_i}$ viewed as an endomorphism of this tensor power of $\bigwedge^{p^i} V_n$ in the natural way. This shows that $W_0 \otimes \cdots \otimes W_r$ is a summand of T. Now let $f : \bigwedge^k V_n \hookrightarrow W_0 \otimes \cdots \otimes W_r$ be the canonical inclusion and $g : W_0 \otimes \cdots \otimes W_r \twoheadrightarrow \bigwedge^k V_n$ be the canonical projection. Over any field, the composition $g \circ f$ is $k!/k_0!(k_1 p)! \cdots (k_r p^r)!$ times the identity endomorphism. Since we are in characteristic p, this scalar is 1 by Lucas' theorem. This shows that f is a split injection, so $\bigwedge^k V_n$ is a summand of $W_0 \otimes \cdots \otimes W_r$, hence, of T. $\square$

Unlike the functor Ψ_n considered in (3.1), the functor Φ_n is neither full nor dense. Nevertheless, Lemma 3.4 implies that

$$\widetilde{\Phi}_n := Q \circ \Phi_n : \mathrm{Kar}(\mathcal{OB}(t_0,\dots,t_r)) \to \overline{\mathcal{T}ilt(G_n)} \tag{3.4}$$

is dense. Moreover, and this is the key step in our argument, $\widetilde{\Phi}_n$ is also full. This assertion will be justified in §4; see Theorem 4.17 (the proof is rather short but there are lots of preliminaries!). Given this fact, we can then apply Lemma 2.6 to see that $\widetilde{\Phi}_n$ descends to a symmetric monoidal equivalence

$$\overline{\Phi}_n : \overline{\mathrm{Kar}(\mathcal{OB}(t_0,\dots,t_r))} \to \overline{\mathcal{T}ilt(G_n)}. \tag{3.5}$$

The equivalence Ξ_n appearing in the Main Theorem may then be obtained by composing $\overline{\Phi}_n$ with a quasi-inverse of the equivalence $\overline{\Psi}_{n_0,\dots,n_r}$ from Lemma 3.3. To complete the proof of the Main Theorem, it just remains to identify the labelings of the irreducible objects; this will be explained in §5.

4. Webs and the Schur category

In this section, we show that the functor $\widetilde{\Phi}_n$ from (3.4) is full. The proof depends ultimately on a result of Donkin [D1, Proposition 3.11], which is a version of skew Howe duality for the general linear group. We will explain this using a diagrammatic rather than algebraic formalism, viewing the Schur algebra in terms of a version of the web category from [CKM]. However, we start from the classical perspective as in [G1].

A *composition* $\lambda \vDash d$ is a finite sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative integers summing to d. We call it a *strict composition* and instead write $\lambda \vDash_s d$ if all of its parts are non-zero. We write $\ell(\lambda)$ for the total number n of parts. There is a right action of S_d on the set of d-tuples of positive integers by place permutation: for $\boldsymbol{i} = (i_1, \dots, i_d)$ and $g \in S_d$ the d-tuple $\boldsymbol{i} \cdot g$ has rth entry $i_{g(r)}$. For $\lambda \vDash d$, the set

$$\mathrm{I}_\lambda := \big\{\boldsymbol{i} = (i_1, \dots, i_d) \,\big|\, \#\{r = 1, \dots, d \,|\, i_r = i\} = \lambda_i \text{ for all } i \in \{1, \dots, \ell(\lambda)\}\big\} \tag{4.1}$$

of all d-tuples with λ_1 entries equal to 1, λ_2 entries equal to 2, and so on, is a single orbit under this action.

For $\lambda, \mu \vDash d$, the symmetric group S_d acts diagonally on the right on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$. The orbits are parametrized by the set $\mathrm{Mat}_{\lambda,\mu}$ of all $\ell(\lambda) \times \ell(\mu)$ matrices with non-negative integer entries such that the entries in the ith row sum to λ_i and the entries in the jth column sum to μ_j for all $i \in \{1, \dots, \ell(\lambda)\}$ and $j \in \{1, \dots, \ell(\mu)\}$. For $A = (a_{i,j}) \in \mathrm{Mat}_{\lambda,\mu}$, the corresponding S_d-orbit on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$ is

$$\Pi_A := \left\{ (\boldsymbol{i}, \boldsymbol{j}) \in \mathrm{I}_\lambda \times \mathrm{I}_\mu \;\middle|\; \begin{array}{l} \#\{r = 1, \dots, d \,|\, (i_r, j_r) = (i, j)\} = a_{i,j} \\ \text{for all } i \in \{1, \dots, \ell(\lambda)\}, j \in \{1, \dots, \ell(\mu)\} \end{array} \right\}. \tag{4.2}$$

For compositions $\lambda, \mu, \nu \vDash d$, $A \in \mathrm{Mat}_{\lambda,\mu}$, $B \in \mathrm{Mat}_{\mu,\nu}$ and $C \in \mathrm{Mat}_{\lambda,\nu}$, define

$$Z(A, B, C) := \#\big\{\boldsymbol{j} \,\big|\, (\boldsymbol{i}, \boldsymbol{j}) \in \Pi_A \text{ and } (\boldsymbol{j}, \boldsymbol{k}) \in \Pi_B\big\}, \tag{4.3}$$

where $(\boldsymbol{i}, \boldsymbol{k})$ is some choice of an element of Π_C. This is well-defined independent of the choice of $(\boldsymbol{i}, \boldsymbol{k})$.

Lemma 4.1. *In the notation of (4.3), suppose that* $(\boldsymbol{i}, \boldsymbol{j}) \in \Pi_A$ *and* $(\boldsymbol{j}, \boldsymbol{k}) \in \Pi_B$ *satisfy* $\mathrm{Stab}_{S_d}(\boldsymbol{i}) \cap \mathrm{Stab}_{S_d}(\boldsymbol{k}) = \mathrm{Stab}_{S_d}(\boldsymbol{j})$. *Then* $Z(A, B, C) = 1$ *if* $(\boldsymbol{i}, \boldsymbol{k}) \in \Pi_C$, *and* $Z(A, B, C) = 0$ *otherwise.*

Proof. Pick $(\boldsymbol{i}', \boldsymbol{k}') \in \Pi_C$. To calculate $Z(A, B, C)$, we need to count the number of $\boldsymbol{j}'$ such that $(\boldsymbol{i}', \boldsymbol{j}') \in \Pi_A$ and $(\boldsymbol{j}', \boldsymbol{k}') \in \Pi_B$. Equivalently, this is the number of $\boldsymbol{j}'$ such that $(\boldsymbol{i}', \boldsymbol{j}') \sim (\boldsymbol{i}, \boldsymbol{j})$ and $(\boldsymbol{j}', \boldsymbol{k}') \sim (\boldsymbol{j}, \boldsymbol{k})$.

If such a $\boldsymbol{j}'$ exists, we can find $g \in S_d$ such that $\boldsymbol{j}' \cdot g = \boldsymbol{j}$, then have that $(\boldsymbol{i}' \cdot g, \boldsymbol{j}) \sim (\boldsymbol{i}, \boldsymbol{j})$ and $(\boldsymbol{j}, \boldsymbol{k}' \cdot g) \sim (\boldsymbol{j}, \boldsymbol{k})$. So there is $h \in \mathrm{Stab}_{S_d}(\boldsymbol{j})$ such that $\boldsymbol{i}' \cdot g = \boldsymbol{i} \cdot h$ and $\boldsymbol{k}' \cdot g = \boldsymbol{k} \cdot h$. As $\mathrm{Stab}_{S_d}(\boldsymbol{j}) \subseteq \mathrm{Stab}_{S_d}(\boldsymbol{i}) \cap \mathrm{Stab}_{S_d}(\boldsymbol{k})$, we deduce that $\boldsymbol{i}' \cdot g = \boldsymbol{i}$ and $\boldsymbol{k}' \cdot g = \boldsymbol{k}$, hence, $(\boldsymbol{i}, \boldsymbol{k}) \in \Pi_C$.

Finally assume that $(\boldsymbol{i}, \boldsymbol{k}) \in \Pi_C$. Then, we may as well assume that $(\boldsymbol{i}', \boldsymbol{k}') = (\boldsymbol{i}, \boldsymbol{k})$, and $Z(A, B, C)$ is the number of $\boldsymbol{j}'$ such that $(\boldsymbol{i}, \boldsymbol{j}') \sim (\boldsymbol{i}, \boldsymbol{j})$ and $(\boldsymbol{j}', \boldsymbol{k}) \sim (\boldsymbol{j}, \boldsymbol{k})$. Any such $\boldsymbol{j}'$ can be written as $\boldsymbol{j} \cdot g$ for $g \in \mathrm{Stab}_{S_d}(\boldsymbol{i}) \cap \mathrm{Stab}_{S_d}(\boldsymbol{k})$. As $\mathrm{Stab}_{S_d}(\boldsymbol{i}) \cap \mathrm{Stab}_{S_d}(\boldsymbol{k}) \subseteq \mathrm{Stab}_{S_d}(\boldsymbol{j})$, we deduce that $\boldsymbol{j}' = \boldsymbol{j}$. This shows that $Z(A, B, C) = 1$. □

The numbers $Z(A, B, C)$ arise naturally as the *structure constants* for multiplication in the Schur algebra. To recall this, let V_n be the defining representation of G_n with standard basis $v_1, \dots, v_n$. The symmetric group S_d acts on the right on the tensor space $V_n^{\otimes d}$ by permuting tensors. The *Schur algebra* is the endomorphism algebra

$$S(n, d) := \mathrm{End}_{S_d}(V_n^{\otimes d}). \tag{4.4}$$

The action of S_d on $V_n^{\otimes d}$ commutes with the action of G_n, hence, it leaves the weight spaces of $V_n^{\otimes d}$ invariant. The weights which arise are the ones in the set

$$\Lambda(n,d) := \{\lambda \vDash d \mid \ell(\lambda) = n\}. \tag{4.5}$$

We deduce that the projection 1_λ of $V_n^{\otimes d}$ onto its λ-weight space gives an idempotent in the Schur algebra. These so-called *weight idempotents* for all $\lambda \in \Lambda(n,d)$ are mutually orthogonal and sum to the identity in $S(n,d)$. Note also that $1_\lambda V_n^{\otimes d}$ has basis $\{v_{\boldsymbol{i}} := v_{i_1} \otimes \cdots \otimes v_{i_d} \mid \boldsymbol{i} \in \mathrm{I}_\lambda\}$, with the action of $g \in S_d$ on this basis satisfying

$$v_{\boldsymbol{i}} g = v_{\boldsymbol{i} \cdot g}. \tag{4.6}$$

For $\lambda, \mu \in \Lambda(n,d)$ and $A \in \mathrm{Mat}_{\lambda,\mu}$, define the linear map

$$\xi_A : 1_\mu V_n^{\otimes d} \to 1_\lambda V_n^{\otimes d}, \qquad v_{\boldsymbol{j}} \mapsto \sum_{\boldsymbol{i} \text{ with } (\boldsymbol{i},\boldsymbol{j}) \in \Pi_A} v_{\boldsymbol{i}}. \tag{4.7}$$

The endomorphisms $\{\xi_A \mid A \in \mathrm{Mat}_{\lambda,\mu}\}$ give *Schur's basis* for $1_\lambda S(n,d) 1_\mu$. Moreover, multiplication in the Schur algebra satisfies

$$\xi_A \circ \xi_B := \sum_{C \in \mathrm{Mat}_{\lambda,\nu}} Z(A,B,C) \xi_C \tag{4.8}$$

for $A \in \mathrm{Mat}_{\lambda,\mu}$ and $B \in \mathrm{Mat}_{\mu,\nu}$. This is *Schur's product rule*; e.g., see [G1, 2.3b].

The algebra $S(n,d)$ can also be constructed starting from the general linear group G_n; see [G1, Ch. 2]. From this approach, one sees that the category $S(n,d)$-mod is identified with the full subcategory of $\mathcal{R}ep(G_n)$ consisting of the *polynomial representations of degree d*. Another important aspect of the theory needed later is the *Schur functor*

$$\pi : S(n,d)\text{-mod} \to \Bbbk S_d\text{-mod} \tag{4.9}$$

as in [G1, Ch. 6]. In Green's approach, this is defined only when $n \geq d$, so that the composition $\omega := (1^d, 0^{n-d})$ belongs to $\Lambda(n,d)$. There is an algebra isomorphism

$$\Bbbk S_d \xrightarrow{\sim} 1_\omega S(n,d) 1_\omega, \qquad g \mapsto \xi_A \tag{4.10}$$

where $A \in \mathrm{Mat}_{\omega,\omega}$ is the $n \times n$ matrix with $a_{g(1),1} = \cdots = a_{g(d),d} = 1$ and all other entries zero. Identifying $\Bbbk S_d$ with $1_\omega S(n,d) 1_\omega$ in this way, π is the idempotent truncation functor associated to the weight idempotent 1_ω. Note also that there is an isomorphism of $(S(n,d), \Bbbk S_d)$-bimodules

$$V_n^{\otimes d} \xrightarrow{\sim} S(n,d) 1_\omega, \qquad v_{\boldsymbol{i}} \mapsto \xi_A \tag{4.11}$$

where A here is the $n \times n$ matrix with $a_{i_1,1} = \cdots = a_{i_d,d} = 1$ and all other entries zero. It follows that the Schur functor π is isomorphic to $\mathrm{Hom}_{G_n}(V_n^{\otimes d}, -)$.

Definition 4.2. The *Schur category* is the strict monoidal category $\mathcal{S}chur$ with

- objects that are all strict compositions $\lambda \vDash_s d$ for all $d \geq 0$;
- for $\lambda \vDash_s d$ and $\mu \vDash_s d'$, the morphism space $\mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda)$ is zero unless $d = d'$, and it is the vector space with basis $\{\xi_A \mid A \in \mathrm{Mat}_{\lambda,\mu}\}$ if $d = d'$;
- the tensor product of objects is defined by concatenation $\lambda \otimes \mu := \lambda \sqcup \mu$;
- the tensor product of morphisms is defined by $\xi_A \otimes \xi_B := \xi_{\mathrm{diag}(A,B)}$, where $\mathrm{diag}(A,B)$ is the obvious block diagonal matrix;
- vertical composition of morphisms is defined by Schur's product rule as in (4.8).

We leave it to the reader to check that the axioms of a strict monoidal category are satisfied. The unit object $\mathbb{1}$ is the composition of length zero, and the identity endomorphism 1_λ of an object $\lambda \in \mathcal{S}chur$ is $\xi_{\mathrm{diag}(\lambda_1,\dots,\lambda_{\ell(\lambda)})}$.

Remark 4.3. Assuming that $n \geq d$, let $\Lambda(n,d)_L$ be the set of compositions $\lambda \in \Lambda(n,d)$ that are *left-justified*, meaning that $\lambda = (\lambda_1, \dots, \lambda_m, 0^{n-m})$ with $\lambda_1, \dots, \lambda_m > 0$. Let $e := \sum_{\lambda \in \Lambda(n,d)_L} 1_\lambda \in S(n,d)$. Any weight idempotent in $S(n,d)$ is conjugate to a left-justified one, hence, the algebras $S(n,d)$ and $eS(n,d)e$ are Morita equivalent. Moreover, there is an obvious algebra isomorphism

$$eS(n,d)e = \bigoplus_{\lambda,\mu \in \Lambda(n,d)_L} 1_\lambda S(n,d) 1_\mu \cong \bigoplus_{\lambda,\mu \vDash_s d} \mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda). \tag{4.12}$$

This makes the connection between the Schur algebra and the Schur category precise.

Remark 4.4. By (4.12) and [FS, Theorem 3.2], the category $\mathcal{S}chur\text{-mod}_{\mathrm{fd}}$ of globally finite-dimensional $\mathcal{S}chur$-modules, i.e., the category of functors $V : \mathcal{S}chur \to \mathcal{V}ec$ such that $\bigoplus_{\lambda \in \mathcal{S}chur} V(\lambda)$ is finite-dimensional, is equivalent to the category $\mathcal{P}ol$ of (strict) *polynomial functors* from [FS]. Under this equivalence, the projective $\mathcal{S}chur$-module $\mathrm{Hom}_{\mathcal{S}chur}((n), -)$ corresponds to the nth divided power functor Γ^n. The category of polynomial functors is symmetric monoidal with a biexact tensor product functor $- \otimes -$ (see e.g. [FS, Proposition 2.6]). This structure can also be seen directly on $\mathcal{S}chur\text{-mod}_{\mathrm{fd}}$ in terms of an induction functor extending the tensor product on the underlying monoidal category $\mathcal{S}chur$. In fact, $\mathcal{P}ol$ is the Abelian envelope of the Karoubian monoidal category $\mathcal{S}chur$ in a precise sense: any functor $F : \mathcal{S}chur \to \mathcal{A}$ to an Abelian category $\mathcal{A}$ factors through the embedding $\mathcal{S}chur \to \mathcal{P}ol$, $Z \mapsto \mathrm{Hom}(Z, -)^*$ to induce a right-exact functor $\mathcal{P}ol \to \mathcal{A}$, which is monoidal in case F is monoidal.

There are some special families of morphisms ξ_A in the Schur category which are easy to understand.

- If A is a $1 \times n$ row matrix, we call ξ_A an *n-fold merge*; the reason for the terminology will become clear when we switch to the diagrammatic formalism below. By Schur's product rule, we have in the Schur category that

$$\xi_{(\,\lambda_1\ \cdots\ \lambda_n\,)} = \xi_{(\,\lambda_1+\cdots+\lambda_m\ \ \lambda_{m+1}+\cdots+\lambda_n\,)} \circ \left(\xi_{(\,\lambda_1\ \cdots\ \lambda_m\,)} \otimes \xi_{(\,\lambda_{m+1}\ \cdots\ \lambda_n\,)}\right) \tag{4.13}$$

  for $\lambda_1, \dots, \lambda_n > 0$ and $1 \leq m < n$; cf. (4.38) below. Using this formula recursively, it follows that any n-fold merge can be expressed as a composition of tensor products of two-fold merges $\xi_{(\,a\ b\,)}$.
- If A is an $n \times 1$ column matrix, we call ξ_A an *n-fold split*. By the analogous (in fact, transpose) formula to (4.13), in the Schur category, any n-fold split can be expressed as a composition of tensor products of two-fold splits $\xi_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$.
- If A is an $n \times n$ monomial matrix, i.e., it has exactly one non-zero entry in every row and column, we call ξ_A a *generalized permutation*. Letting λ and μ be the row and column sums of A, so that $A \in \mathrm{Mat}_{\lambda,\mu}$, we may also use the notation

$$1_\lambda g = g 1_\mu := \xi_A \tag{4.14}$$

  where $g \in S_n$ is defined from $\lambda = g(\mu)$; here we are using the left action of S_n on $\Lambda(n,d)$ so $g(\mu) = (\mu_{g^{-1}(1)}, \dots, \mu_{g^{-1}(n)})$. In other words, g is the permutation such that $a_{g(1),1} = \mu_1, \dots, a_{g(n),n} = \mu_n$. Given another permutation $h \in S_n$, Schur's product rule implies that

$$1_\lambda(gh) = g1_\mu \circ 1_\mu h = (gh)1_\nu \tag{4.15}$$

  for $\mu = h(\nu)$. This may also be deduced as a special case of the following lemma.

Lemma 4.5. *Suppose that $A \in \mathrm{Mat}_{\lambda,\mu}$ and $B \in \mathrm{Mat}_{\mu,\nu}$ for $\lambda, \mu, \nu \vDash_s d$. Assume:*

- *A has a unique non-zero entry in every column, so that there is an associated function $\alpha : \{1, \dots, \ell(\mu)\} \to \{1, \dots, \ell(\lambda)\}$ sending i to the unique j such that $a_{j,i} \neq 0$;*

- *B has a unique non-zero entry in every row, so that there is an associated function $\beta : \{1, \dots, \ell(\mu)\} \to \{1, \dots, \ell(\nu)\}$ sending i to the unique j such that $b_{i,j} \neq 0$;*
- *the function $\gamma : \{1, \dots, \ell(\mu)\} \to \{1, \dots, \ell(\lambda)\} \times \{1, \dots, \ell(\nu)\}, i \mapsto (\alpha(i), \beta(i))$ is injective.*

Then $\xi_A \circ \xi_B = \xi_C$ where $C \in \mathrm{Mat}_{\lambda,\nu}$ is the matrix with $c_{\alpha(i),\beta(i)} = \mu_i$ for $i \in \{1, \dots, \ell(\mu)\}$, all other entries being zero.

Proof. Let $n := \ell(\mu)$ and $\boldsymbol{j} := (1^{\mu_1}, 2^{\mu_2}, \dots, n^{\mu_n})$. Let $\boldsymbol{i} := \alpha(\boldsymbol{j})$ and $\boldsymbol{k} := \beta(\boldsymbol{j})$, i.e., these are the tuples obtained by applying the functions α and β to the entries of $\boldsymbol{j}$. Then we have that $(\boldsymbol{i}, \boldsymbol{j}) \in \Pi_A, (\boldsymbol{j}, \boldsymbol{k}) \in \Pi_B$, and $(\boldsymbol{i}, \boldsymbol{k}) \in \Pi_C$. Moreover, the injectivity of γ implies that $\mathrm{Stab}_{S_d}(\boldsymbol{i}) \cap \mathrm{Stab}_{S_d}(\boldsymbol{k}) = \mathrm{Stab}_{S_d}(\boldsymbol{j})$. Now apply Lemma 4.1. □

Now suppose that $A \in \mathrm{Mat}_{\lambda,\mu}$ for $\lambda, \mu \vDash_s d$.

- Let A^- be the block diagonal matrix $\mathrm{diag}(A_1, \dots, A_{\ell(\lambda)})$ where A_i is the $1 \times n_i$ matrix obtained from the ith row of A by removing all entries 0. Note that
$$\xi_{A^-} = \xi_{A_1} \otimes \cdots \otimes \xi_{A_{\ell(\lambda)}}, \tag{4.16}$$
with each ξ_{A_i} being an n_i-fold merge. Also let λ^- be the composition recording the column sums of A^-, so that $A^- \in \mathrm{Mat}_{\lambda,\lambda^-}$. The ith entry λ_i^- of λ^- is the ith non-zero entry of the sequence $a_{1,1}, a_{1,2}, \dots, a_{1,\ell(\mu)}, a_{2,1}, \dots$ that is the *row reading* of the matrix A.
- Let A^+ be the block diagonal matrix $\mathrm{diag}(A^1, \dots, A^{\ell(\mu)})$ where A^i is the $n^i \times 1$ matrix obtained from the ith column of A by removing all entries 0. We then have that
$$\xi_{A^+} = \xi_{A^1} \otimes \cdots \otimes \xi_{A^{\ell(\mu)}}, \tag{4.17}$$
with each ξ_{A^i} being an n^i-fold split. Also let μ^+ be the composition recording the row sums of A^+, so that $A^+ \in \mathrm{Mat}_{\mu^+,\mu}$. The ith entry μ_i^+ of μ^+ is the ith non-zero entry of the sequence $a_{1,1}, a_{2,1}, \dots, a_{\ell(\lambda),1}, a_{1,2}, \dots$ that is the *column reading* of A.
- The composition λ^- is a rearrangement of μ^+, in particular, $n := \ell(\lambda^-) = \ell(\mu^+)$. Let $f_1 : \{1, \dots, n\} \to \{1, \dots, \ell(\lambda)\}$ and $f_2 : \{1, \dots, n\} \to \{1, \dots, \ell(\mu)\}$ be defined so that λ_i^-, the ith non-zero entry of the row reading of A, is in row $f_1(i)$ and column $f_2(i)$. Let $h_1 : \{1, \dots, n\} \to \{1, \dots, \ell(\lambda)\}$ and $h_2 : \{1, \dots, n\} \to \{1, \dots, \ell(\mu)\}$ be defined so that μ_i^+, the ith non-zero entry of the column reading of A, is in row $h_1(i)$ and column $h_2(i)$. There is then a unique permutation $g \in S_n$ such that $(f_1(g(i)), f_2(g(i))) = (h_1(i), h_2(i))$ for each $i \in \{1, \dots, n\}$. We have in particular that $g(\mu^+) = \lambda^-$. Let $A^\circ \in \mathrm{Mat}_{\lambda^-,\mu^+}$ be the $n \times n$ monomial matrix with $(g(i), i)$-entry equal to μ_i^+ for $i = 1, \dots, n$, all other entries being zero. We have that
$$\xi_{A^\circ} = g 1_{\mu^+}, \tag{4.18}$$
notation as in (4.14).

For example, suppose that $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \end{pmatrix}$, so $\lambda = (4, 5)$ and $\mu = (3, 2, 4)$. Then

$$A^- = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}, \quad A^\circ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.19}$$

Also $\lambda^- = (1, 3, 2, 2, 1)$ and $\mu^+ = (1, 2, 2, 3, 1)$, so that $\xi_{A^\circ} = g 1_{\mu^+}$ where $g = (2\,3\,4)$; see also (4.37) below for a helpful picture of this situation.

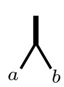
Lemma 4.6. *For $A \in \mathrm{Mat}_{\lambda,\mu}$, we have that $\xi_A = \xi_{A^-} \circ \xi_{A^\circ} \circ \xi_{A^+}$.*

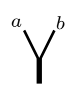
Proof. Define n, λ^-, μ^+ and f_1, f_2, g, h_1, h_2 as above. First, we apply Lemma 4.5 with $\alpha = g$ and $\beta = h_2$ to deduce that $\xi_{A^\circ} \circ \xi_{A^+} = \xi_B$ for $B \in \mathrm{Mat}_{\lambda^-,\mu}$ defined so that $b_{g(i),h_2(i)} = \mu_i^+$ for $i = 1, \dots, n$, all other entries being zero. Then apply it again with $\alpha = f_1$ and $\beta = g \circ h_2$ to show that $\xi_{A^-} \circ \xi_B = \xi_A$. $\square$

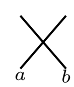
Lemma 4.6 shows that any ξ_A can be expressed as the vertical composition of some tensor product of merges, a generalized permutation, and some tensor product of splits. This statement is very natural from the diagrammatic point of view which we are going to explain next.

In fact, we are going to prove that $\mathcal{S}chur$ is isomorphic to a version of the web category from [CKM, §5][2] for polynomial representations of the general linear group, but in the stable limit as the rank tends to infinity. This stable version, which is well known to the experts, is easier than the finite rank version in [CKM] since one can exploit the connection to $\mathcal{S}chur$ and the defining basis for morphism spaces in the latter category. We will explain this in detail below since it is hard to extract from the existing literature. See also Remark 4.15 which explains how to recover the finite rank cases (together with a natural basis for their morphism spaces) via this approach.

Definition 4.7. The *polynomial web category* $\mathcal{W}eb$ is the strict monoidal category defined by generators and relations as follows. Its objects are all strict compositions with tensor product being by concatenation as in Definition 4.2. The one-part compositions (a) for $a > 0$ give a family of generating objects. In string diagrams, we will represent the generating object (a) as a string labeled by the *thickness* a, and a general object $\lambda = (\lambda_1, \dots, \lambda_n)$ will be a sequence of strings of thicknesses $\lambda_1, \dots, \lambda_n > 0$ in order from left to right. Then there are generating morphisms

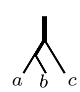


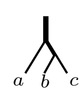
$: (a,b) \to (a+b)$,


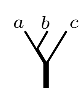
$: (a+b) \to (a,b)$,


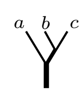
$: (a,b) \to (b,a)$ (4.20)

for $a, b > 0$, which we call the two-fold merge, the two-fold split, and the thick crossing, respectively. The generating morphisms are subject to the following relations for $a, b, c, d > 0$ with $d - a = c - b$:

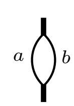


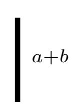
$=$


$,$


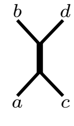
$=$


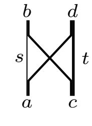
$,$ (4.21)



$= \binom{a+b}{a}$


$,$ (4.22)



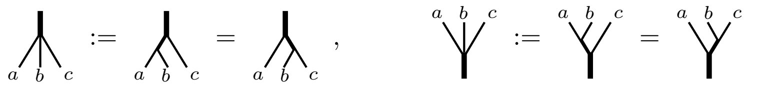
$= \sum_{\substack{0 \le s \le \min(a,b) \\ 0 \le t \le \min(c,d) \\ t-s=d-a}}$


$.$ (4.23)

In diagrams for morphisms in $\mathcal{W}eb$, we often omit thickness labels on strings when they are implicitly determined by the other labels. We have not defined any morphisms that could be drawn as cups or caps, so the strings in these diagrams have singular points where crossings and splits/merges occur, but no critical points of slope zero.

[2]This extended work of G. Kuperberg to whom the reference to spiders is credited.

The relation (4.21) means that we can introduce more general *n-fold merges* and *n-fold splits* for $n \geq 2$ by composing the two-fold ones in an obvious way (cf. (4.13)). For example, the three-fold merges and splits are defined from

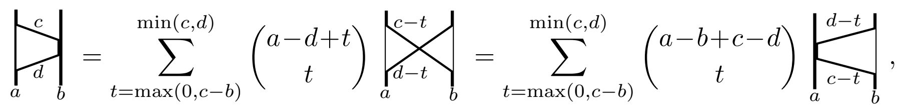

(4.24)

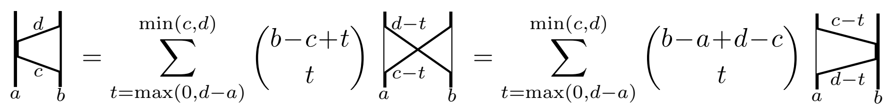
By the symmetry of Definition 4.7, there are isomorphisms of strict monoidal categories

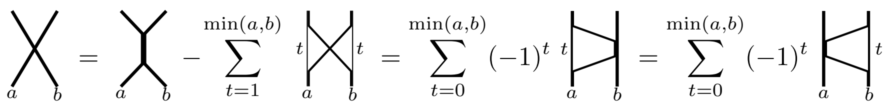
$$\mathtt{T} : \mathcal{W}eb \to \mathcal{W}eb^{\mathrm{op}}, \qquad\qquad \mathtt{R} : \mathcal{W}eb \to \mathcal{W}eb^{\mathrm{rev}} \tag{4.25}$$

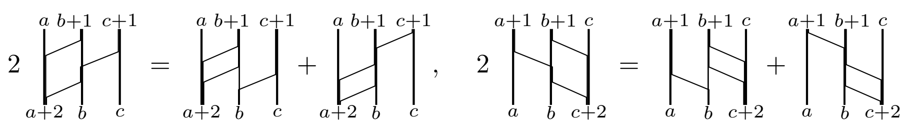
defined by reflecting diagrams in a horizontal or vertical axis, respectively.

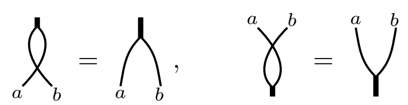
We will need various other relations which are consequences of the defining relations. The proofs of these are elementary relation chases and will be explained in the appendix.

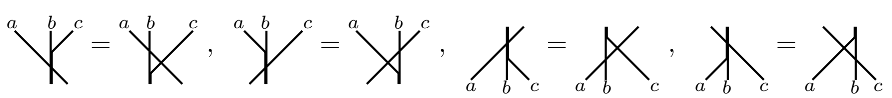

(4.26)

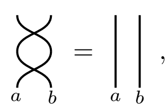

(4.27)

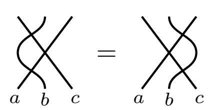

(4.28)


(4.29)


(4.30)

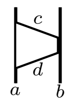

(4.31)

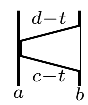

(4.32)


(4.33)

The relations (4.31)–(4.33) imply that $\mathcal{W}eb$ has the structure of a strict symmetric monoidal category, with symmetric braiding defined on generating objects by the thick crossings.

Remark 4.8. In view of (4.28), the thick crossings can be expressed in terms of the two-fold merges and splits, so they are redundant as generators. In fact, as will also be proved in the appendix, $\mathcal{W}eb$ is isomorphic to the strict monoidal category with generators that are just the two-fold merges and splits, subject to the relations (4.21) and (4.22) as before together with the *square switch relations*



$$= \sum_{t=\max(0,c-b)}^{\min(c,d)} \binom{a-b+c-d}{t} \quad (4.34)$$


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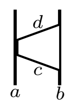
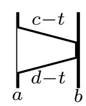


$$= \sum_{t=\max(0,d-a)}^{\min(c,d)} \binom{b-a+d-c}{t} \quad (4.35)$$


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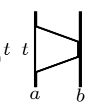
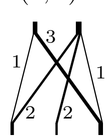
which are as in (4.26)–(4.27) above. This is the original presentation from [CKM, §5], where the square switch relations are interpreted in terms of the commutator relation between the divided powers $e_i^{(c)}, f_i^{(d)}$. From this perspective, the relations (4.29) come from the Serre relations. Then the thick crossings get *defined* from the formula

$$:= \sum_{t=0}^{\min(a,b)} (-1)^t \quad (4.36)$$


,

which is [CKM, Corollary 6.2.3] (up to multiplication by the sign $(-1)^{ab}$ which also appears in the statement of Theorem 4.14 below). In [CKM], this formula is explained in terms of the action of the ith simple reflection on the appropriate weight space of a polynomial representation of GL_n: $s_i = e_i^{(b)} f_i^{(a)} - e_i^{(b-1)} f_i^{(a-1)} + \cdots$.

For $\lambda, \mu \vDash_s d$, a $\lambda \times \mu$ *chicken foot diagram*[3] is a diagram representing a morphism in $\mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$ in which the thick strings determined by μ at the bottom of the diagram split into thinner strings, then these thinner strings cross each other in some way in the middle of the diagram, before merging back into the thick strings determined by λ at the top. This means that a chicken foot diagram has three distinct parts, the top and bottom parts which consist just of merges and splits, respectively, all of which occur at the same horizontal level, and the middle part which is a generalized permutation diagram. Here is an example with $\lambda = (4, 5)$ and $\mu = (3, 2, 4)$:



. (4.37)

We say that a chicken foot diagram is *reduced* if there is at most one intersection or join between every pair of the thinner strings in the diagram. Thus, for each $i \in \{1, \dots, \ell(\lambda)\}$ and $j \in \{1, \dots, \ell(\mu)\}$, there is at most one string connecting the ith vertex at the top to the jth vertex at the bottom, and moreover the generalized permutation diagram in the middle of the diagram corresponds to a reduced word in the symmetric group. The *type* of a reduced chicken foot diagram is the matrix $A \in \mathrm{Mat}_{\lambda,\mu}$ whose (i, j)-entry is the thickness of the unique string connecting the ith vertex at the top to the jth vertex at the bottom, or zero if there is no such string. For example, (4.37) is a reduced chicken foot diagram of type $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{\lambda,\mu}$, and the top, middle and bottom parts of (4.37) are reduced chicken foot diagrams whose types are given by the matrices A^-, A^0 and A^+ from (4.19).

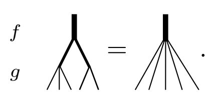
[3]This terminology was suggested to the first author by A. Kleshchev.

By the braid relations (4.33), all reduced chicken foot diagrams of the same type $A \in \mathrm{Mat}_{\lambda,\mu}$ represent the same morphism $[A] \in \mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$. In fact, we are going to prove that these morphisms for all $A \in \mathrm{Mat}_{\lambda,\mu}$ give a basis for space $\mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$. The fact that they span is established in the next lemma, which gives a straightening algorithm to convert an arbitrary diagram for a morphism in $\mathcal{W}eb$ into a linear combination of reduced chicken foot diagrams.

Lemma 4.9. *The morphism space* $\mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$ *is spanned by the morphisms* $[A]$ *for all* $A \in \mathrm{Mat}_{\lambda,\mu}$.

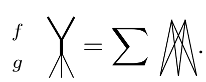
Proof. We have observed already that $\mathcal{W}eb$ is generated by its two-fold merges and splits. Since these are themselves defined by reduced chicken foot diagrams, it suffices to show for any morphism f that consists of a two-fold merge or a two-fold split (tensored on the left and right by appropriate identity morphisms), and any morphism g defined by a reduced $\lambda \times \mu$ chicken foot diagram, that the vertical composition $f \circ g$ can be expressed as a linear combination of reduced chicken foot diagrams.

Suppose first that f involves a two-fold merge joining to the ith and $(i+1)$th strings at the top of g. If g has an r-fold merge at its ith vertex and an s-fold merge at its $(i+1)$th vertex, then we can use (4.21) to rewrite $f \circ g$ so that it is a $\lambda' \times \mu$ chicken foot diagram with an $(r+s)$-fold merge at its ith vertex, where λ' is the composition $(\lambda_1, \dots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{\ell(\lambda)})$. For example:


(4.38)

However the resulting chicken foot diagram is not necessarily reduced. It remains to observe that the morphism defined by a non-reduced chicken foot diagram can be converted to a scalar multiple of a morphism defined by a reduced one just using the relations (4.21)–(4.22) and (4.30)–(4.33).

Now suppose that f involves a two-fold split joining to the ith vertex at the top of g. Say this vertex of g involves an n-fold merge. Using (4.21), (4.23) and (4.31), we rewrite the composition of the split in f and this merge in g as a sum of reduced chicken foot diagrams. For example:


(4.39)

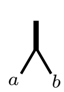
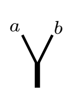
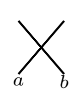
Then compose these diagrams with the remainder of the diagram, using (4.31) then (4.21) again to commute the splits at the bottom of this part of the resulting diagrams downwards past the generalized permutation part of g. $\square$

Theorem 4.10. *There is an isomorphism of strict monoidal categories*

$$F : \mathcal{W}eb \xrightarrow{\sim} \mathcal{S}chur$$

which is the identity on objects (i.e., strict compositions) and sends the morphism $[A] \in \mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$ *defined by a reduced chicken foot diagram of type* $A \in \mathrm{Mat}_{\lambda,\mu}$ *to Schur's basis element* $\xi_A \in \mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda)$. *In particular, the functor* F *sends the generating morphisms (4.20) to the two-fold merge* $\xi_{(\,a\ b\,)}$, *the two-fold split* $\xi_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$ *and the generalized permutation* $\xi_{\left(\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right)}$, *respectively.*

Proof. We define F to be the identity on objects, and define it on the generating morphisms for $\mathcal{W}eb$ so that

 $\mapsto \xi_{(\,a\ b\,)}$,  $\mapsto \xi_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$,  $\mapsto \xi_{\left(\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right)}$.

To see that this is well-defined, we just need to verify that the defining relations (4.21)–(4.23) of $\mathcal{W}eb$ are satisfied in $\mathcal{S}chur$. This is an application of Schur's product rule; in particular, (4.21) for merges follows by the identity (4.13) already checked above.

Now take $A \in \mathrm{Mat}_{\lambda,\mu}$. The morphism $[A] \in \mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$ is the vertical concatenation $[A^-] \circ [A^\circ] \circ [A^+]$ for A^-, A° and A^+ defined prior to Lemma 4.6. This follows because the reduced chicken foot diagrams for A^-, A° and A^+ give the top, middle and bottom parts of the one for A. From (4.13) (and its analog for splits) and (4.16)–(4.17), it follows that $F([A^-]) = \xi_{A^-}$ and $F([A^+]) = \xi_{A^+}$. Also $[A^\circ]$ is a generalized permutation, so by (4.18) we have that $F([A^\circ]) = \xi_{A^\circ}$. It remains to apply Lemma 4.6 to deduce that $F([A]) = \xi_A$.

Since the morphisms ξ_A for $A \in \mathrm{Mat}_{\lambda,\mu}$ form a basis for $\mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda)$ by Definition 4.2, and the corresponding morphisms $[A]$ span $\mathrm{Hom}_{\mathcal{W}eb}(\mu, \lambda)$ by Lemma 4.9, we deduce that F is full and faithful. Hence, it is an isomorphism. □

From now on, we will *identify* the categories $\mathcal{W}eb$ and $\mathcal{S}chur$ via the isomorphism F from Theorem 4.10. We will refer to this category as the Schur category rather than the polynomial web category, and will not use the notation $\mathcal{W}eb$ again.

Remark 4.11. The Schur algebra possesses another classical basis, namely, Green's basis of codeterminants; see [G2, W]. Using Remark 4.3, it is straightforward to translate Green's result to obtain another basis for the morphism space $\mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda)$, as follows. Suppose that $\lambda, \mu \vDash_s d$. For a partition $\kappa \vdash d$, let $\mathrm{Std}(\lambda, \kappa)$ denote the set of all semistandard Young tableaux of shape κ and content λ, i.e., fillings of the Young diagram of κ with λ_1 entries equal to 1, λ_2 entries equal to 2, . . . , so that the entries are weakly increasing along rows and strictly decreasing down columns. Define $\mathrm{Std}(\mu, \kappa)$ similarly. For $P \in \mathrm{Std}(\lambda, \kappa)$ and $Q \in \mathrm{Std}(\mu, \kappa)$, let

$$\gamma_{P,Q} := \xi_A \circ \xi_B \tag{4.40}$$

where $A \in \mathrm{Mat}_{\lambda,\kappa}$ (resp., $B \in \mathrm{Mat}_{\kappa,\mu}$) is defined so that $a_{i,j}$ is the number of entries i in the jth row of P (resp., $b_{i,j}$ is the number of entries j in the ith row of Q). Note that a reduced chicken foot diagram of type A has no merges, while one of type B has no splits. Consequently, the diagram for $\gamma_{P,Q}$ can look rather different than a chicken foot diagram: it has generalized permutations at the top and bottom and merges and splits in the middle. The *codeterminant basis* for $\mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda)$ is

$$\{\gamma_{P,Q} \mid d \geq 0, \kappa \vdash d, P \in \mathrm{Std}(\lambda, \kappa), Q \in \mathrm{Std}(\mu, \kappa)\}. \tag{4.41}$$

This basis is of a similar nature to the basis recently constructed from a completely different viewpoint by Elias [E]. It gives $\mathcal{S}chur$ the structure of an object-adapted cellular category in the sense of [EL, Definition 2.1].

It is time to return to the study of the category $\mathcal{T}ilt(G_n)$ of tilting modules for G_n. For $\lambda \vDash d$, let

$$\textstyle\bigwedge^\lambda V_n := \bigwedge^{\lambda_1} V_n \otimes \cdots \otimes \bigwedge^{\lambda_{\ell(\lambda)}} V_n \in \mathcal{T}ilt(G_n). \tag{4.42}$$

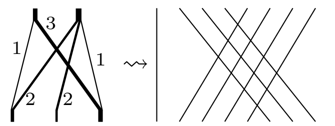
Let S_λ denote the standard parabolic subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_{\ell(\lambda)}}$ of the symmetric group S_d. Given also $\mu \vDash d$, let $(S_\lambda \backslash S_d)_{\min}$ and $(S_d/S_\mu)_{\min}$ be the sets of minimal length $S_\lambda \backslash S_d$- and S_d/S_μ-coset representatives, respectively. Then

$$(S_\lambda \backslash S_d/S_\mu)_{\min} := (S_\lambda \backslash S_d)_{\min} \cap (S_d/S_\mu)_{\min}$$

is the set of minimal length $S_\lambda \backslash S_d/S_\mu$-double coset representatives, and there is a bijection

$$\mathrm{Mat}_{\lambda,\mu} \xrightarrow{\sim} (S_\lambda \backslash S_d/S_\mu)_{\min}, \qquad A \mapsto d_A. \tag{4.43}$$

To construct d_A from A, take a reduced chicken foot diagram of type A; for once, we are not assuming λ and μ are strict here, so A may have rows or columns of zeros, in which case we mean the same diagram as for the matrix obtained from A by removing these trivial rows and columns. Then expand this diagram by replacing each string of thickness r by r parallel strings of unit thickness. The desired double coset representative d_A is the element of S_d defined by the resulting permutation diagram. For example, for A as in (4.37), the diagram expands as



and $d_A = (2\,5\,8\,4\,7\,3\,6)$.

Lemma 4.12. *Suppose $\lambda, \mu \vDash d$ and $A \in \mathrm{Mat}_{\lambda,\mu}$. We have that $d_A^{-1} S_\lambda d_A \cap S_\mu = S_{\mu^+}$ for some $\mu^+ \vDash d$ (see the discussion after (4.17) for an explicit construction of μ^+). There is a unique G_n-module homomorphism ϕ_A making the diagram*

$$\begin{tikzcd} V_n^{\otimes d} \arrow[r] \arrow[d] & V_n^{\otimes d} \arrow[d] \\ \bigwedge^\mu V_n \arrow[r, "\phi_A"] & \bigwedge^\lambda V_n \end{tikzcd}$$

commute, where the top map is the G_n-module homomorphism defined by right multiplication by $\sum_{g \in (S_\mu/S_{\mu^+})_{\min}} (-1)^{\ell(g d_A^{-1})} g d_A^{-1} \in \Bbbk S_d$, and the vertical maps are the natural quotients.

Proof. The first statement follows from [DJ, Lemma 1.6(ii)]. The kernel of the projection $V_n^{\otimes d} \twoheadrightarrow \bigwedge^\mu V_n$ is spanned by the fixed point sets of the involutions of $V_n^{\otimes d}$ defined by right multiplication by all simple reflections $s \in S_\mu$. Thus, to complete the proof, we need to show for such an s and $v \in V_n^{\otimes d}$ with $vs = v$ that the vector

$$w := \sum_{g \in (S_\mu/S_{\mu^+})_{\min}} (-1)^{\ell(g d_A^{-1})} v g d_A^{-1}$$

is in the kernel of the projection $V_n^{\otimes d} \twoheadrightarrow \bigwedge^\lambda V_n$. For $g \in (S_\mu/S_{\mu^+})_{\min}$, we either have that $sgS_{\mu^+} \neq gS_{\mu^+}$, in which case $sg \in (S_\mu/S_{\mu^+})_{\min}$ too, or $sgS_{\mu^+} = gS_{\mu^+}$, in which case $g^{-1}sg \in S_{\mu^+}$; see [DJ, Lemma 1.1]. It follows that $(S_\mu/S_{\mu^+})_{\min}$ decomposes as $X \sqcup sX \sqcup Y$ such that $\ell(sx) = \ell(x) + 1$ for all $x \in X$, and $y^{-1}sy \in S_{\mu^+}$ for all $y \in Y$. For $x \in X$, we have that $(-1)^{\ell(xd_A^{-1})} vxd_A^{-1} + (-1)^{\ell(sxd_A^{-1})} vsxd_A^{-1} = 0$ as $vs = v$. This implies that

$$w = \sum_{y \in Y} (-1)^{\ell(yd_A^{-1})} vyd_A^{-1}.$$

It remains to show for $y \in Y$ that vyd_A^{-1} is in the kernel of $V_n^{\otimes d} \twoheadrightarrow \bigwedge^\lambda V_n$. We have that $syd_A^{-1} = yd_A^{-1}t$ for $t := d_A(y^{-1}sy)d_A^{-1} \in S_\lambda$. By [DJ, Lemma 1.6(iv)], $\ell(yd_A^{-1}t) = \ell(y) + \ell(d_A^{-1}) + \ell(t)$. Since $\ell(syd_A^{-1}) \leq \ell(y) + \ell(d_A^{-1}) + 1$, we deduce that $\ell(t) = 1$. Moreover $vyd_A^{-1}t = vsyd_A^{-1} = vyd_A^{-1}$. This shows that vyd_A^{-1} is a fixed point for the simple reflection $t \in S_\lambda$, thus, it is in the kernel of the projection. $\square$

Proposition 4.13 (Donkin)**.** *Fix integers $m, d \geq 0$. For any $n \geq 0$, there is a surjective algebra homomorphism*

$$f_n : S(m,d) \twoheadrightarrow \mathrm{End}_{G_n}\left(\bigoplus_{\lambda \in \Lambda(m,d)} \bigwedge^\lambda V_n \right) \tag{4.44}$$

sending $\xi_A \in 1_\lambda S(m,d) 1_\mu$ to the endomorphism that is equal to the homomorphism ϕ_A from Lemma 4.12 on the summand $\bigwedge^\mu V_n$, and is zero on all other summands. Moreover, f_n is an isomorphism if $n \geq d$.

Proof. This is proved in [D1], but we need to go through the argument in detail in order to identify the map f_n explicitly. We just treat the case that $n \geq d$. Then the existence and surjectivity of f_n for $n < d$ follows from the existence and surjectivity of f_N for $N \geq d$ by an argument involving truncation to the subgroup $G_n < G_N$. This step is explained in the proof of [D1, Proposition 3.11]; it depends on [D1, Proposition 1.5], hence, on homological properties arising from the fact that Schur algebras are quasi-hereditary algebras.

So now assume that $n \geq d$. We must show that f_n is a well-defined algebra isomorphism. For $\lambda \in \Lambda(m,d)$, let $M(\lambda)$ be the right permutation module $X_\lambda \otimes_{\Bbbk S_\lambda} \Bbbk S_d$, where X_λ is the trivial one-dimensional right S_λ-module with generator x_λ. The module $M(\lambda)$ is isomorphic to the λ-weight space $1_\lambda V_m^{\otimes d}$ of $V_m^{\otimes d}$ via the unique S_d-module homomorphism sending $x_\lambda \otimes 1 \in M(\lambda)$ to $v_1^{\otimes \lambda_1} \otimes \cdots \otimes v_m^{\otimes \lambda_m}$. By the definition (4.4) (with n replaced by m of course), we have that

$$S(m,d) \cong \mathrm{End}_{S_d}\left(\bigoplus_{\lambda \in \Lambda(m,d)} M(\lambda)\right).$$

Under this isomorphism, $\xi_A \in 1_\lambda S(m,d) 1_\mu$ corresponds to the unique S_d-module homomorphism $M(\mu) \to M(\lambda)$ sending $x_\mu \otimes 1$ to $x_\lambda \otimes \sum_{g \in (S_{\mu^+} \backslash S_\mu)_{\min}} d_A g$, where $S_{\mu^+} = d_A^{-1} S_\lambda d_A \cap S_\mu$ as in Lemma 4.12. This follows from (4.7), noting that $S_{\mu^+} = \mathrm{Stab}_{S_d}(\boldsymbol{i} \cdot d_A) \cap \mathrm{Stab}_{S_d}(\boldsymbol{j})$ where $\boldsymbol{i} = (1^{\lambda_1}, \dots, m^{\lambda_m})$ and $\boldsymbol{j} = (1^{\mu_1}, \dots, m^{\mu_m})$.

Consider instead the left signed permutation module $N(\lambda) := \Bbbk S_d \otimes_{\Bbbk S_\lambda} Y_\lambda$, where Y_λ is the one-dimensional left S_λ-module with generator y_λ such that $g y_\lambda = (-1)^{\ell(g)} y_\lambda$ for all $g \in S_\lambda$. Noting that $N(\lambda)$ is isomorphic to $M(\lambda)$ tensored by sign and converted from a right module to a left module using the antiautomorphism $g \mapsto g^{-1}$, we deduce from the previous paragraph that there is an algebra isomorphism

$$S(m,d) \cong \mathrm{End}_{S_d}\left(\bigoplus_{\lambda \in \Lambda(m,d)} N(\lambda)\right).$$

Under this isomorphism, $\xi_A \in 1_\lambda S(m,d) 1_\mu$ corresponds to the unique S_d-module homomorphism $N(\mu) \to N(\lambda)$ sending $1 \otimes y_\mu$ to $\sum_{g \in (S_\mu / S_{\mu^+})_{\min}} (-1)^{\ell(g d_A^{-1})} g d_A^{-1} \otimes y_\lambda$.

Now we are going to apply the Schur functor π from (4.9). The key observation is that $\pi\left(\bigwedge^\lambda V_n\right) \cong N(\lambda)$, there being a unique such isomorphism sending the canonical image of $v_1 \otimes \cdots \otimes v_d$ in $\bigwedge^\lambda V_n$ to $1 \otimes y_\lambda$. The head of $\bigwedge^\mu V_n$ and the socle of $\bigwedge^\lambda V_n$ are p-restricted in the sense that they only involve irreducible modules which are not annihilated by π. Indeed, these modules are both submodules and quotient modules of the tensor space $V_n^{\otimes d}$, which has p-restricted head and socle by [BK, Corollary 2.12]. Consequently, by [BK, Lemma 2.17(ii)] (another well-known property of Schur functors), the Schur functor induces an isomorphism

$$\mathrm{Hom}_{S(n,d)}(\textstyle\bigwedge^\mu V_n, \bigwedge^\lambda V_n) \xrightarrow{\sim} \mathrm{Hom}_{S_d}(N(\mu), N(\lambda));$$

see also [D1, Lemma 3.6]. It follows that π induces an algebra isomorphism

$$\mathrm{End}_{G_n}\left(\bigoplus_{\lambda \in \Lambda(m,d)} \textstyle\bigwedge^\lambda V_n\right) \cong \mathrm{End}_{S_d}\left(\displaystyle\bigoplus_{\lambda \in \Lambda(m,d)} N(\lambda)\right).$$

Composing this with the isomorphism in the previous paragraph gives the desired isomorphism f_n.

It just remains to identify the endomorphism $f_n(\xi_A)$ with ϕ_A. For this, it suffices to check for $\xi_A \in 1_\lambda S(m,d) 1_\mu$ that the maps $f_n(\xi_A)$ and ϕ_A are equal on the canonical image of $v_1 \otimes \cdots \otimes v_d$ in $\bigwedge^\mu V_n$. By the definition from Lemma 4.12, ϕ_A sends this vector to the canonical image of

$$\sum_{g \in (S_\mu / S_{\mu^+})_{\min}} (-1)^{\ell(g d_A^{-1})} (v_1 \otimes \cdots \otimes v_d) g d_A^{-1}$$

in $\bigwedge^\lambda V_n$. On the other hand, by the construction of f_n from the previous two paragraphs, $f_n(\xi_A)$ takes this vector to the image of

$$\sum_{g \in (S_\mu / S_{\mu^+})_{\min}} (-1)^{\ell(g d_A^{-1})} g d_A^{-1} (v_1 \otimes \cdots \otimes v_d),$$

where $g d_A^{-1} \in S_d$ is being identified with an element of $1_\omega S(n,d) 1_\omega$ via the isomorphism (4.10). It remains to observe for $g \in S_d$ that $g(v_1 \otimes \cdots \otimes v_d) = (v_1 \otimes \cdots \otimes v_d) g$. This follows because the isomorphism (4.11) maps $v_1 \otimes \cdots \otimes v_d$ to 1_ω. $\square$

The following theorem gives a reformulation of Proposition 4.13 from the perspective of the Schur category.

Theorem 4.14. *There is a full monoidal functor $\Sigma_n : \mathcal{S}chur \to \mathcal{T}ilt(G_n)$ sending an object $\lambda \vDash_s d$ to $\bigwedge^\lambda V_n \in \mathcal{T}ilt(G_n)$, and a morphism ξ_A for $\lambda, \mu \vDash_s d$ and $A \in \mathrm{Mat}_{\lambda,\mu}$ to the homomorphism $\phi_A : \bigwedge^\mu V_n \to \bigwedge^\lambda V_n$ from Lemma 4.12. In particular, Σ_n maps the two-fold merge from (4.20) to the projection $\bigwedge^a V_n \otimes \bigwedge^b V_n \twoheadrightarrow \bigwedge^{a+b} V_n$, the two-fold split to the inclusion*

$$\bigwedge^{a+b} V_n \hookrightarrow \bigwedge^a V_n \otimes \bigwedge^b V_n,$$

$$v_{i_1} \wedge \cdots \wedge v_{i_{a+b}} \mapsto \sum_{g \in (S_{a+b}/S_a \times S_b)_{\min}} (-1)^{\ell(g)} v_{i_{g(1)}} \wedge \cdots \wedge v_{i_{g(a)}} \otimes v_{i_{g(a+1)}} \wedge \cdots \wedge v_{i_{g(a+b)}},$$

and the thick crossing to the isomorphism $\bigwedge^a V_n \otimes \bigwedge^b V_n \xrightarrow{\sim} \bigwedge^b V_n \otimes \bigwedge^a V_n, v \otimes w \mapsto (-1)^{ab} w \otimes v$.

Proof. To see that Σ_n is a well-defined functor, we need to show that $\Sigma_n(\xi_A \circ \xi_B) = \Sigma_n(\xi_A) \circ \Sigma_n(\xi_B)$ for $A \in \mathrm{Mat}_{\lambda,\mu}$ and $B \in \mathrm{Mat}_{\mu,\nu}$ for $\lambda, \mu, \nu \vDash_s d$ and $d \geq 0$. By Schur's product rule, $\Sigma_n(\xi_A \circ \xi_B) = \sum_{C \in \mathrm{Mat}_{\lambda,\nu}} Z(A,B,C) \Sigma_n(\xi_C) = \sum_{C \in \mathrm{Mat}_{\lambda,\nu}} Z(A,B,C) \phi_C$. We need to show this equals $\phi_A \circ \phi_B$. This follows from Proposition 4.13 and (4.12) with n replaced by $m \geq d$. The proposition also shows that Σ_n is full. Finally, to see that Σ_n is a monoidal functor, we need to check that $\phi_A \otimes \phi_B = \phi_{\mathrm{diag}(A,B)}$. This is clear from the explicit description of these maps given by Lemma 4.12. $\square$

Remark 4.15. The functor Σ_n in Theorem 4.14 is certainly not faithful, but it is *asymptotically faithful* in the sense that it induces an isomorphism

$$\mathrm{Hom}_{\mathcal{S}chur}(\mu, \lambda) \xrightarrow{\sim} \mathrm{Hom}_{G_n}(\bigwedge^\mu V_n, \bigwedge^\lambda V_n) \tag{4.45}$$

for n sufficiently large relative to λ and μ. In fact, if $\lambda, \mu \vDash_s d$ then one just needs that $n \geq d$, as is clear from the last part of Proposition 4.13. Let

$$\mathcal{S}chur_n := \mathcal{S}chur / \mathcal{J}_n \tag{4.46}$$

where $\mathcal{J}_n$ is the tensor ideal of $\mathcal{S}chur$ that is the kernel of Σ_n. Then Σ_n induces an equivalence of symmetric monoidal categories between $\mathcal{S}chur_n$ and the full monoidal subcategory of $\mathcal{T}ilt(G_n)$ generated by the exterior powers $\bigwedge^a V_n$ for all $a > 0$. In fact, $\mathcal{J}_n$ is the tensor ideal of $\mathcal{S}chur$ generated by the morphisms $1_{(m)}$ for all $m > n$; cf. Remark 3.1. Together with Theorem 4.10, this identifies $\mathcal{S}chur_n$ with the polynomial web category for GL_n from [CKM, §5]. This can be seen from [CKM], but also it can be proved quite easily using the codeterminant basis from Remark 4.11, as follows.

Note first that the tensor ideal $\mathcal{K}_n$ of $\mathcal{S}chur$ generated by the morphisms $1_{(m)}$ for all $m > n$ is contained in $\mathcal{J}_n$ as $\bigwedge^m V_n = 0$ for $m > n$. Now take $\lambda, \mu \vDash_s d$. The codeterminants $\gamma_{P,Q}$ for $\kappa \vdash d$ with $\kappa_1 > n$, $P \in \mathrm{Std}(\lambda,\kappa)$ and $Q \in \mathrm{Std}(\mu,\kappa)$ belong to $\mathcal{K}_n(\mu,\lambda)$ since their diagrams involve a string of thickness κ_1. Hence, $\mathrm{Hom}_{\mathcal{S}chur}(\mu,\lambda)/\mathcal{K}_n(\mu,\lambda)$ is spanned by all $\gamma_{P,Q}$ for $\kappa \vdash d$ with $\kappa_1 \leq n$, $P \in \mathrm{Std}(\lambda,\kappa)$ and $Q \in \mathrm{Std}(\mu,\kappa)$. In fact, we have that $\mathcal{K}_n(\mu,\lambda) = \mathcal{J}_n(\mu,\lambda)$ (proving the assertion), and these codeterminants with $\kappa_1 \leq n$ give a basis for $\mathrm{Hom}_{\mathcal{S}chur_n}(\mu,\lambda) \cong \mathrm{Hom}_{G_n}(\bigwedge^\mu V_n, \bigwedge^\lambda V_n)$. This follows because

$$\dim \mathrm{Hom}_{G_n}(\textstyle\bigwedge^\mu V_n, \bigwedge^\lambda V_n) = \# \left\{ (\kappa, P, Q) \;\middle|\; \begin{array}{l} \kappa \vdash d \text{ with } \kappa_1 \leq n, \\ P \in \mathrm{Std}(\lambda,\kappa), Q \in \mathrm{Std}(\mu,\kappa) \end{array} \right\}.$$

(Proof: For $\kappa \vdash d$ with $\kappa_1 \leq n$, let κ^T be the transpose partition viewed as a weight in X_n^+. By the Littlewood-Richardson rule and character considerations, the tilting module $\bigwedge^\mu V_n$ has a Δ-flag with sections $\Delta_n(\kappa^T)$ for all such κ, each appearing with multiplicity $\#\,\mathrm{Std}(\mu,\kappa)$. Similarly $\bigwedge^\lambda V_n$ has a ∇-flag with sections $\nabla_n(\kappa^T)$, each appearing with multiplicity $\#\,\mathrm{Std}(\lambda,\kappa)$. Now use $\dim \mathrm{Ext}^i_{G_n}(\Delta_n(\sigma), \nabla_n(\tau)) = \delta_{\sigma,\tau}\delta_{i,0}$.)

At last, all of the background is in place, and we can achieve the main goal of the section. The composition of the functor Σ_n from Theorem 4.14 with the quotient functor $Q : \mathcal{T}ilt(G_n) \to \overline{\mathcal{T}ilt(G_n)}$ gives us a full monoidal functor

$$\widetilde{\Sigma}_n : \mathcal{S}chur \to \overline{\mathcal{T}ilt(G_n)}. \tag{4.47}$$

We just need one more elementary observation.

Lemma 4.16. *Suppose that $p > 0$ and a, b are positive integers summing to p^m. The images under $\widetilde{\Sigma}_n$ of the two-fold merge and split morphisms from (4.20) are both zero.*

Proof. By weight considerations, $\mathrm{Hom}_{G_n}\left(\bigwedge^{p^m} V_n, \bigwedge^a V_n \otimes \bigwedge^b V_n\right)$ is of dimension one with basis given by the two-fold split. So $\mathrm{Hom}_{\overline{\mathcal{T}ilt(G_n)}}\left(\bigwedge^{p^m} V_n, \bigwedge^a V_n \otimes \bigwedge^b V_n\right)$ is spanned by the image f of the two-fold split. Similarly, the image g of the two-fold merge spans $\mathrm{Hom}_{\overline{\mathcal{T}ilt(G_n)}}\left(\bigwedge^a V_n \otimes \bigwedge^b V_n, \bigwedge^{p^m} V_n\right)$. By semisimplicity, if one of these morphisms is non-zero, so is the other, and $g \circ f$ is an automorphism of $\bigwedge^{p^m} V_n$. But this composition is zero by (4.22). $\square$

Theorem 4.17. *The functor $\widetilde{\Phi}_n : \mathrm{Kar}(\mathcal{OB}(t_0,\dots,t_r)) \to \overline{\mathcal{T}ilt(G_n)}$ from (3.4) is full.*

Proof. Let X and Y be objects of $\mathcal{OB}(t_0,\dots,t_r)$, so they are both words in the symbols $\uparrow_i$'s and $\downarrow_i$'s for $i = 0,\dots,r$. Their images $\overline{X}$ and $\overline{Y}$ under the functor $\widetilde{\Phi}_n$ are corresponding tensor products of the modules $\bigwedge^{p^i} V_n$ and $\bigwedge^{p^i} V_n^*$, notation as in (1.4). We need to show that the linear map

$$\mathrm{Hom}_{\mathcal{OB}(t_0,\dots,t_r)}(X,Y) \to \mathrm{Hom}_{\overline{\mathcal{T}ilt(G_n)}}(\overline{X},\overline{Y})$$

defined by the functor $\widetilde{\Phi}_n$ is surjective. Since this is a symmetric monoidal functor, we may assume that all of the $\downarrow_i$'s in X appear at the beginning of this word. Then using duality we can transfer them from the beginning of X to $\uparrow_i$'s appearing at the beginning of Y. Thus we are reduced to the case that X only involves $\uparrow_i$'s. Repeating the argument for Y, we reduce further to the case that Y only involves $\uparrow_i$'s too.

So now X and Y are words just in the symbols $\uparrow_i$ for $i = 0,\dots,r$, and $\overline{X}$ and $\overline{Y}$ are corresponding tensor products of the modules $\bigwedge^{p^i} V_n$, i.e., we have that $\overline{X} = \bigwedge^\mu V_n$ and $\overline{Y} = \bigwedge^\lambda V_n$ for strict compositions λ,μ all of whose parts are of the form p^i for $i = 0,\dots,r$. Since the functor $\widetilde{\Sigma}_n$ is full, it follows that $\mathrm{Hom}_{\overline{\mathcal{T}ilt(G_n)}}(\overline{X},\overline{Y})$ is spanned by the images of the morphisms ξ_A for $A \in \mathrm{Mat}_{\lambda,\mu}$. In view of Lemma 4.16, these images are zero unless $A = A^\circ$, i.e., ξ_A is a generalized permutation. As generalized permutations are generated by thick crossings, and Σ_n maps thick crossings to tensor flips (up to a sign) according to Theorem 4.14, it remains to observe that the tensor flip

$$\textstyle\bigwedge^{p^i} V_n \otimes \bigwedge^{p^j} V_n \to \bigwedge^{p^j} V_n \otimes \bigwedge^{p^i} V_n, \qquad v \otimes w \mapsto w \otimes v$$

is the image under Φ_n of the crossing in $\mathcal{OB}(t_0, \dots, t_r)$ of strings of color i and j. $\square$

5. Identification of labelings

Let notation be as in (1.4), and recall (1.5)–(1.6). We have now proved the existence of a symmetric monoidal equivalence

$$\Xi_n : \overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})} \to \overline{\mathcal{T}ilt(G_n)} \tag{5.1}$$

sending $V_{n_i} \in \overline{\mathcal{T}ilt(G_{n_i})}$ to $\bigwedge^{p^i} V_n \in \overline{\mathcal{T}ilt(G_n)}$ for $i = 0, \dots, r$. To complete the proof of the Main Theorem, it remains to show that Ξ_n sends $T_{n_0}(\lambda^{(0)}) \boxtimes \cdots \boxtimes T_{n_r}(\lambda^{(r)})$ to $T_n(\imath(\underline{\lambda}))$ for $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r)}) \in X^+_{n_0,p} \times \cdots \times X^+_{n_r,p}$.

Let $\Lambda^+_n \subset X^+_n$ denote the set of polynomial dominant weights, i.e., the weights $\lambda \in \mathbb{Z}^n$ such that $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Let $\Lambda^+_{n,p} := \Lambda^+_n \cap X^+_{n,p}$. Let $\varpi_i = (1^i, 0^{n-i})$ be the highest weight of $\bigwedge^i V_n$ and $\det_n := \bigwedge^n V_n$ be the determinant representation.

Lemma 5.1. *Given $0 \leq k \leq n$, we have that $\binom{n}{k} \not\equiv 0 \pmod p$ if and only if*

$$k = k_0 + k_1 p + \cdots + k_r p^r \quad \text{with} \quad 0 \leq k_i \leq n_i \text{ for all } i = 0, \dots, r. \tag{5.2}$$

Assuming this is the case, the function $\imath$ takes $(\varpi_{k_0}, \dots, \varpi_{k_r}) \in X^+_{n_0} \times \cdots \times X^+_{n_r}$ to $\varpi_k \in X^+_n$. Also the equivalence Ξ_n sends $\bigwedge^{k_0} V_{n_0} \boxtimes \cdots \boxtimes \bigwedge^{k_r} V_{n_r}$ to a copy of $\bigwedge^k V_n$.

Proof. The first statement follows from Lucas' theorem (1.7). Also it is easy to see that $\imath((\varpi_{k_0}, \dots, \varpi_{k_r})) = \varpi_k$ just using the combinatorial definition of $\imath$. For the final assertion, note that $k_i < p$, so $\bigwedge^{k_i} V_{n_i}$ is the summand of $V_{n_i}^{\otimes k_i}$ defined by the idempotent $e_i := \frac{1}{k_i!} \sum_{g \in S_{k_i}} (-1)^{\ell(g)} g \in \Bbbk S_{k_i} = \mathrm{End}_{G_{n_i}}(V_{n_i}^{\otimes k_i})$. So by the definition of the functor, Ξ_n takes $\bigwedge^{k_0} V_{n_0} \boxtimes \cdots \boxtimes \bigwedge^{k_r} V_{n_r} \in \overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})}$ to the tensor product $W_0 \otimes \cdots \otimes W_r \in \overline{\mathcal{T}ilt(G_n)}$ where $W_i \cong \bigwedge^{k_i p^i} V_n$ is the summand of $\left(\bigwedge^{p^i} V_n\right)^{\otimes k_i}$ defined by e_i viewed now as an endomorphism of $\left(\bigwedge^{p^i} V_n\right)^{\otimes k_i}$. In particular, since Ξ_n is an equivalence, this shows that $W_0 \otimes \cdots \otimes W_r$ is an irreducible object in $\overline{\mathcal{T}ilt(G_n)}$. It remains to observe that $W_0 \otimes \cdots \otimes W_r \cong \bigwedge^k V_n$ in $\overline{\mathcal{T}ilt(G_n)}$. This follows because $\bigwedge^k V_n$ is a summand of $W_0 \otimes \cdots \otimes W_r$ in $\mathcal{T}ilt(G_n)$, as we established already in the proof of Lemma 3.4. $\square$

Corollary 5.2. *The equivalence Ξ_n sends $\det_{n_0} \boxtimes \cdots \boxtimes \det_{n_r}$ to $\det_n$.*

Using Corollary 5.2, the problem in hand reduces easily to the case of polynomial weights. To analyze polynomial weights, we need one more observation. For $\lambda \in \Lambda^+_n$, let λ^T be the usual transpose partition. By the definitions, a weight $\lambda \in \Lambda^+_n$ belongs to $\Lambda^+_{n,p}$ if and only if

$$\lambda^T = (\lambda^{(0)})^T + p(\lambda^{(1)})^T + \cdots + p^r(\lambda^{(r)})^T \quad \text{with} \quad \lambda^{(i)} \in \Lambda^+_{n_i,p} \text{ for } i = 0, \dots, r. \tag{5.3}$$

Choose $m \geq \lambda_1$. Then we can view all of the partitions in the decomposition (5.3) as elements of Λ^+_m. Recall that $\lambda \in \Lambda^+_m$ is *p-restricted* if $\lambda_i - \lambda_{i+1} < p$ for each $i = 1, \dots, m-1$. Since $n_i < p$ for each i, the weight $(\lambda^{(i)})^T$ has first part that is smaller than p, so it is certainly p-restricted. We deduce by the Steinberg tensor product theorem that

$$L_m(\lambda^T) \cong L_m((\lambda^{(0)})^T) \otimes L_m((\lambda^{(1)})^T)^{[1]} \otimes \cdots \otimes L_m((\lambda^{(r)})^T)^{[r]}, \tag{5.4}$$

where $[k]$ denotes the kth Frobenius twist. This observation will be used in the proof of the next result.

Theorem 5.3. *For $\lambda \in \Lambda_n^+ \setminus \Lambda_{n,p}^+$, we have that $\dim T_n(\lambda) \equiv 0 \pmod p$. If $\lambda \in \Lambda_{n,p}^+$, so that it is the image under $\imath$ of some $(\lambda^{(0)}, \dots, \lambda^{(r)}) \in \Lambda_{n_0,p}^+ \times \cdots \times \Lambda_{n_r,p}^+$, we have that $T_n(\lambda) \cong \Xi_n\big(T_{n_0}(\lambda^{(0)}) \boxtimes \cdots \boxtimes T_{n_r}(\lambda^{(r)})\big)$ in $\overline{\mathcal{T}ilt(G_n)}$*

Proof. We proceed by induction on the lexicographic ordering on Λ_n^+. The base case $\lambda = 0$ is trivial as Ξ_n sends $\mathbb{1}$ to $\mathbb{1}$. For the induction step, take $0 \neq \lambda \in \Lambda_n^+$ and pick $m \geq \lambda_1$. Let $\mu \in \Lambda_n^+$ be obtained by removing some column of height $0 < k \leq n$ from the Young diagram of λ, i.e., $\lambda = \mu + \varpi_k$. Then $T_n(\lambda)$ is a summand of $\bigwedge^k V_n \otimes T_n(\mu)$. If $\lambda \in \Lambda_{n,p}^+$ then $\mu \in \Lambda_{n,p}^+$ too and k is of the form (5.2). This follows from the combinatorial definition of the function $\imath$. By induction, $\mu \in \Lambda_{n,p}^+$ if and only if $\dim T_n(\mu) \not\equiv 0 \pmod p$.

Suppose that $\dim \bigwedge^k V_n \otimes T_n(\mu) \equiv 0 \pmod p$. Then $\bigwedge^k V_n \otimes T_n(\mu)$ is zero in $\overline{\mathcal{T}ilt(G_n)}$ (as one of the tensor factors is negligible), hence, so is its summand $T_n(\lambda)$. Thus, $\dim T_n(\lambda) \equiv 0 \pmod p$. Using Lemma 5.1 and the observations made at the end of previous paragraph, we also have that $\lambda \notin \Lambda_{n,p}^+$ in this situation, so this is consistent with what we are trying to prove.

Now suppose that $\dim \bigwedge^k V_n \otimes T_n(\mu) \not\equiv 0 \pmod p$. Then we can write k as $k_0 + k_1 p + \cdots + k_r p^r$ as in (5.2) and μ^T as $(\mu^{(0)})^T + \cdots + p^r(\mu^{(r)})^T$ as in (5.3). Note also that $\mu_1 \leq m-1$, so that we can view μ^T and all $(\mu^{(i)})^T$ here as elements of Λ_{m-1}^+. By [BK, Theorem B(ii)], we have that

$$\bigwedge^k V_n \otimes T_n(\mu) \cong T_n(\lambda) \oplus \bigoplus_{\lambda \rhd \nu \in \Lambda_n^+} T_n(\nu)^{\oplus\left[L_m(\nu^T)_k : L_{m-1}(\mu^T)\right]},$$

where for a G_m-module M we write M_k for the sum of its weight spaces for all weights with mth coordinate equal to k, viewing this as a module over the naturally embedded subgroup G_{m-1}. By induction, $T_n(\nu)$ is zero in $\overline{\mathcal{T}ilt(G_n)}$ unless $\nu \in \Lambda_{n,p}^+$. So we deduce in $\overline{\mathcal{T}ilt(G_n)}$ that

$$\bigwedge^k V_n \otimes T_n(\mu) \cong T_n(\lambda) \oplus \bigoplus_{\lambda \rhd \nu \in \Lambda_{n,p}^+} T_n(\nu)^{\oplus\left[L_m(\nu^T)_k : L_{m-1}(\mu^T)\right]}. \tag{5.5}$$

Each ν here can be decomposed as $(\nu^{(0)})^T + \cdots + p^r(\nu^{(r)})^T$ according to (5.3), and then we can use the Steinberg decomposition (5.4) to see that

$$\left[L_m(\nu^T)_k : L_{m-1}(\mu^T)\right] = \prod_{i=0}^{r} \left[L_m((\nu^{(i)})^T)_{k_i} : L_{m-1}((\mu^{(i)})^T)\right]. \tag{5.6}$$

Now we apply [BK, Theorem B(ii)] again to see that

$$\bigwedge^{k_i} V_{n_i} \otimes T_{n_i}(\mu^{(i)}) \cong T_{n_i}(\lambda^{(i)}) \oplus \bigoplus_{\lambda^{(i)} \rhd \nu^{(i)} \in \Lambda_{n_i,p}^+} T_{n_i}(\nu^{(i)})^{\oplus[L_m((\nu^{(i)})^T)_k : L_{m-1}((\mu^{(i)})^T)]}$$

in $\overline{\mathcal{T}ilt(G_{n_i})}$, where $\lambda^{(i)} := \mu^{(i)} + \varpi_{k_i} \in \Lambda_{n_i}^+$, i.e., its Young diagram is obtained from the one for $\mu^{(i)}$ by adding a column of height k_i (we do not claim here that $\lambda^{(i)} \in \Lambda_{n_i,p}^+$

necessarily). We deduce from this isomorphism for all $i = 0, \dots, r$ plus (5.6) that

$$\left(\bigwedge^{k_0} V_{n_0} \otimes T_{n_0}(\mu^{(0)})\right) \boxtimes \cdots \boxtimes \left(\bigwedge^{k_r} V_{n_r} \otimes T_{n_r}(\mu^{(r)})\right) \cong T_{n_0}(\lambda^{(0)}) \boxtimes \cdots \boxtimes T_{n_r}(\lambda^{(r)})$$
$$\oplus \bigoplus_{\mu \rhd \nu \in \Lambda^+_{n,p}} \left(T_{n_0}(\nu^{(0)}) \boxtimes \cdots \boxtimes T_{n_r}(\nu^{(r)})\right)^{\oplus \left[L_m(\nu^T)_k : L_{m-1}(\mu^T)\right]} \tag{5.7}$$

in $\overline{\mathcal{T}ilt(G_{n_0})} \boxtimes \cdots \boxtimes \overline{\mathcal{T}ilt(G_{n_r})}$, for $\nu^{(i)}$ defined from $\nu^T = (\nu^{(0)})^T + p(\nu^{(1)})^T + \cdots + p^r(\nu^{(r)})^T$ again. Now we apply the monoidal functor Ξ_n to (5.7) using Lemma 5.1 and the induction hypothesis. Comparing the result with (5.5) and using semisimplicity shows finally that

$$\Xi_n \left(T_{n_0}(\lambda^{(0)}) \boxtimes \cdots \boxtimes T_{n_r}(\lambda^{(r)})\right) \cong T_n(\lambda)$$

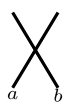
in $\overline{\mathcal{T}ilt(G_n)}$. In particular, $\dim T_n(\lambda) \equiv 0 \pmod p$ unless $\lambda^{(i)} \in \Lambda^+_{n_i,p}$ for all $i = 0, \dots, r$. Since λ is μ with a column of height k added and $\lambda^{(i)}$ is $\mu^{(i)}$ with a column of height k_i added, the weight λ is the image of $(\lambda^{(0)}, \dots, \lambda^{(r)})$ under $\imath$. The induction step now follows from this isomorphism. $\square$

Theorem 5.3 and Corollary 5.2 together complete the proof of the Main Theorem.

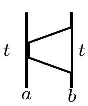
Appendix A. Relations

In this appendix, we prove the relations formulated in §4.

To start with, we explain how to deduce (4.23) from the relations (4.21)–(4.22) and the square switch relations (4.34)–(4.35), interpreting thick crossings as the morphisms defined by (4.36). Note for this that, in the presence of the square switch relations, the definition (4.36) is equivalent to

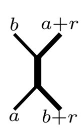


$$:= \sum_{t=0}^{\min(a,b)} (-1)^t$$

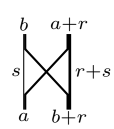


$$. \tag{A.1}$$

This is an easy exercise. Now let notation be as in (4.23) and set $r := d - a$. We just treat the case $r \geq 0$; the other case $r \leq 0$ then follows by reflecting in a vertical axis and using (A.1). We must prove that



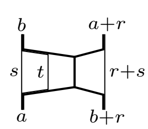
$$= \sum_{s=0}^{\min(a,b)}$$



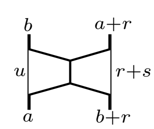
$$, \tag{A.2}$$

We first substitute the definition (4.36) into the right hand side of (A.2), using (4.21)–(4.22), to get

$$\sum_{s=0}^{\min(a,b)} \sum_{t=0}^{\min(a,b)-s} (-1)^t$$



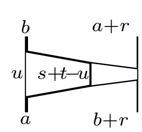
$$= \sum_{s=0}^{\min(a,b)} \sum_{u=s}^{\min(a,b)} (-1)^{s+u} \binom{u}{s}$$



$$. \tag{A.3}$$

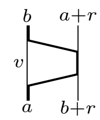
Then we square switch to see that this equals

$$\sum_{s=0}^{\min(a,b)} \sum_{u=s}^{\min(a,b)} \sum_{t=u-s}^{\min(a-s,b-s)} (-1)^{s+u} \binom{u}{s} \binom{u+r}{t}$$

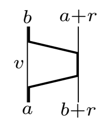


.

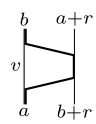
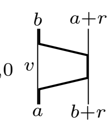
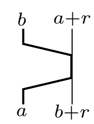
Using (4.21)–(4.22) again, this simplifies to

$$\sum_{s=0}^{\min(a,b)} \sum_{u=s}^{\min(a,b)} \sum_{v=u}^{\min(a,b)} (-1)^{s+u} \binom{u}{s} \binom{u+r}{v-s} \binom{v}{u}$$  .

Next, switch the orders of the summations to get

$$\sum_{v=0}^{\min(a,b)} (-1)^v \sum_{s=0}^{v} (-1)^s \left(\sum_{u=s}^{v} (-1)^{u-v} \binom{u}{s} \binom{v}{u} \binom{u+r}{v-s} \right)$$  .

The term in parentheses is equal to $\binom{v}{s}$; to see this, take the identity from Lemma A.1, replace m, n, r and s with $s+r, v-s, u-s$ and $v-u$, respectively, then multiply both sides by $\binom{v}{s}$. Hence, we have

$$\sum_{v=0}^{\min(a,b)} (-1)^v \left(\sum_{s=0}^{v} (-1)^s \binom{v}{s} \right)$$  $$= \sum_{v=0}^{\min(a,b)} (-1)^v \delta_{v,0}$$  $=$  ,

which is the left hand side of (A.2).

Lemma A.1. *Let $\binom{m}{r,s}$ be the trinomial coefficient $m(m-1)\cdots(m-r-s+1)/r!s!$ (interpreted as 0 if $r<0$ or $s<0$). For $m \in \mathbb{Z}$ and $n \geq 0$, we have that*

$$\sum_{r+s=n} (-1)^s \binom{m+r}{r, s} = 1.$$

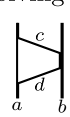
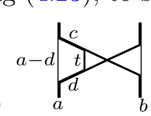
Proof. Use the recurrence relation $\binom{m}{r,s} = \binom{m-1}{r,s} + \binom{m-1}{r-1,s} + \binom{m-1}{r,s-1}$ and induction on n to show that

$$\sum_{r+s=n} (-1)^s \binom{m+r}{r, s} = \sum_{r+s=n} (-1)^s \binom{m-1+r}{r, s}.$$

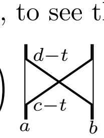
Hence, we may assume that $m = 0$, when the identity is clear. ☐

In the remainder of the appendix, we work in the category $\mathcal{W}eb$ as defined in Definition 4.7, so have the defining relations (4.21)–(4.23), and will prove the relations (4.26)–(4.33). In particular, this shows that the relations (4.21)–(4.23) imply the square switch relations, justifying the equivalence of presentations asserted in Remark 4.8.

Proof of (4.26). Note $a \geq d$. To prove the first equality, we expand the left hand side as a sum of diagrams involving a crossing using (4.23), to see that

 $$= \sum_{t=\max(0,c-b)}^{\min(c,d)}$$  .

Then use (4.21)–(4.22). A similar argument establishes the first equality in (4.27). Then to prove the second equality in (4.26), we use the first equality from (4.27) to expand the right hand side, with the variable t replaced by u, to see that it equals

$$\sum_{u=\max(0,c-b)}^{\min(c,d)} \sum_{t=u}^{\min(c,d)} \binom{a-b+c-d}{u} \binom{b-c+t}{t-u}$$  .

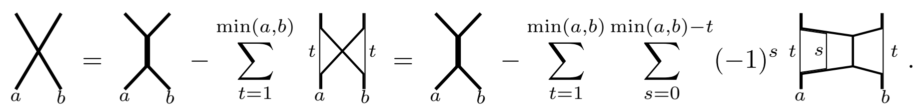
Now switch the summations and use the standard binomial coefficient identity

$$\sum_{u=0}^{t} \binom{a-b+c-d}{u}\binom{b-c+t}{t-u} = \binom{a-d+t}{t}.$$

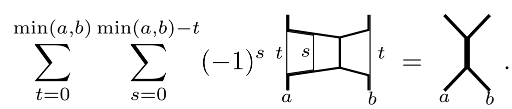
(Proof: Compute x^t-coefficients in $(1+x)^{a-b+c-d}(1+x)^{b-c+t} = (1+x)^{a-d+t}$ in two different ways.)

Proof of (4.27). This follows by reflecting (4.26) in a vertical axis.

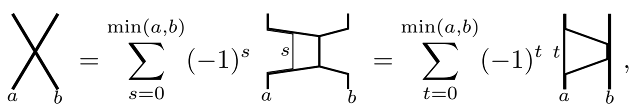
Proof of (4.28). The first equality is immediate from the $r=0$ case of (4.23). Also the final equality follows from the middle one on reflecting in a vertical axis. It remains to establish the middle one. For this, we proceed by induction on $a+b$. The base case $a=b=1$ reduces to the first equality. For the induction step, we have by the first equality and the induction hypothesis that



We saw a similar expression to this before in (A.3); we showed there just using the relations (4.21)–(4.22) and the square switch relations established now by (4.26)–(4.27) that



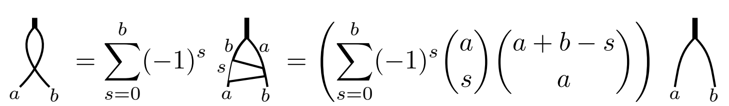
Thus, we have shown that



as required.

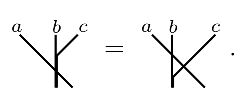
Proof of (4.29). This is explained in the proof of [CKM, Lemma 2.2.1] (and actually plays no role in this article).

Proof of (4.30). By reflection, we just need to prove the first equality, and moreover we may assume that $a \geq b$. Replacing the crossing with (4.36) then using (4.21)–(4.22) as usual, we have that

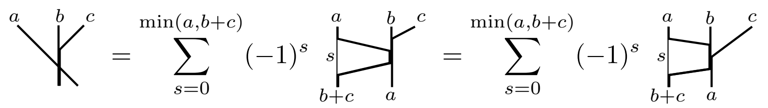


It remains to observe that the coefficient here equals 1. This follows by Lemma A.1, taking $m := a$ and $n := b$.

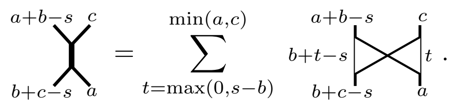
Proof of (4.31). Note the four identities are all equivalent upon reflection, so we just prove the first one:



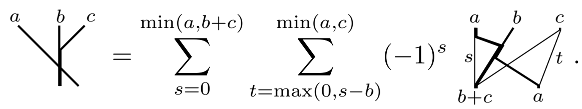
We proceed by induction on $a+b+c$. The base case is when $a=0$, which is trivial. For the induction step, notice that the diagram on the right hand side is a reduced chicken foot diagram. The idea is to expand the left hand side in terms of reduced chicken foot diagrams too, then the equality will be apparent. First we rewrite the crossing at the bottom of this diagram using (4.28):



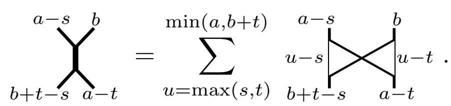
By (4.23), we have that



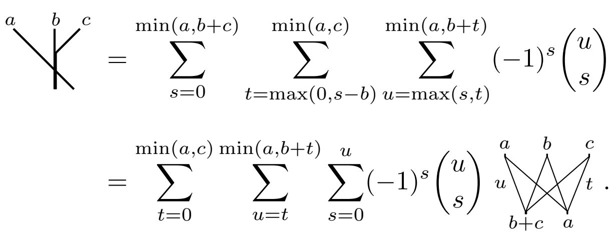
We substitute this into our formula to obtain



By (4.23) again, we have that

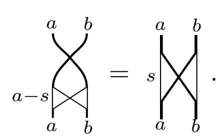


Using this, (4.21)–(4.22), and the induction hypothesis to pull a two-fold split past the string of thickness $c - t$, we simplify further to get

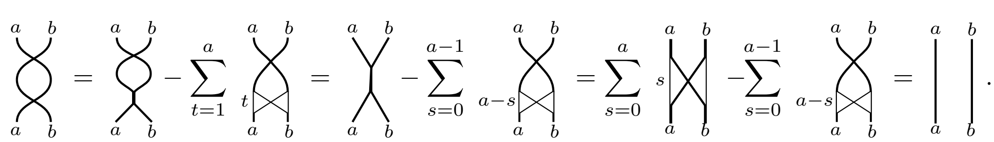


Since $\sum_{s=0}^{u}(-1)^s\binom{u}{s} = \delta_{u,0}$, which is zero unless $u = 0$, when it is 1, the only non-zero term arises when $u = t = 0$, and we get exactly the right hand side we were after.

Proof of (4.32). We proceed by induction on $a+b$, the case $a+b = 1$ being trivial. For the induction step, we may assume without loss of generality that $a \leq b$. We claim for $0 \leq s < a$ that



To see this, one uses (4.30)–(4.31) plus the induction hypothesis to pull the two-fold merges past the crossing. Using the claim, (4.23) and (4.28)–(4.30), we deduce that



Proof of (4.33). Replace the crossing of the strings of thickness a, b on both sides with (4.36). Then use (4.31)–(4.32) to pull the string of thickness c past this expansion of the crossing.

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