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BOOK REVIEW

Finite Dimensional Algebras and Quantum Groups, by Bangming Deng, Jie Du, Brian Parshall and Jianpan Wang, Mathematical Surveys and Monographs, 150, American Mathematical Society, Providence, RI, 2008, xxvi + 759 pp., $119, hardcover, ISBN 13: 978-0-8218-4186-0.

The last thirty years has seen dramatic progress in representation theory due in part to the introduction of some remarkable bases, first for the group algebras of Weyl groups, and second for the universal enveloping algebras of semisimple Lie algebras. These bases are the *Kazhdan-Lusztig bases* from [7], and Lusztig's *canonical bases* originating in [10] (which are Kashiwara's *global crystal bases*), respectively. Their construction relies crucially on certain deformed versions of the underlying algebras, namely, *Iwahori-Hecke algebras* of Weyl groups, and the Drinfeld-Jimbo *quantized enveloping algebras*.

Iwahori-Hecke algebras and Kazhdan-Lusztig polynomials. Suppose we are given a Dynkin diagram Γ with vertices labelled $1, \dots, n$. Let W be the corresponding Weyl group with simple reflections $s_1, \dots, s_n$. For example, if Γ is the Dynkin diagram A_n then W is the symmetric group on $(n+1)$ letters, and one can take s_i to be the ith basic transposition $(i\ \ i{+}1)$. The finite group W is a Coxeter group generated by $s_1, \dots, s_n$ subject only to the relations

$$s_i^2 = 1, \qquad \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ times}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ times}} \tag{1}$$

where m_{ij} is $2, 3, 4$ or 6 according to whether the ith and jth vertices in the Dynkin diagram are joined by zero, one, two or three bonds. Any $w \in W$ can be written as a product $w = s_{i_1} \cdots s_{i_m}$ of simple reflections; if m is minimal then $s_{i_1} \cdots s_{i_m}$ is a *reduced expression* for w and $\ell(w) := m$ is the *length* of w.

Now let $q > 1$ be a prime power and G_q be the group of $\mathbb{F}_q$-rational points in some split reductive algebraic group with underlying Dynkin diagram Γ. Let B_q be a Borel subgroup of G_q and set $X_q := G_q/B_q$. For example, if $\Gamma = A_n$ as before, we could take $G_q = GL_{n+1}(\mathbb{F}_q)$, and X_q is the finite set of all full flags in the vector space $(\mathbb{F}_q)^{n+1}$. We are interested in the *permutation module* $\mathbb{C}X_q$, the module for the group algebra $\mathbb{C}G_q$ that arises from the natural action of G_q on X_q. Its endomorphism algebra is the *Iwahori-Hecke algebra*

$$H_q := \operatorname{End}_{\mathbb{C}G_q}(\mathbb{C}X_q),$$

which plays a fundamental role in the character theory of the finite group G_q. It was first described explicitly by Iwahori [5], who called it the "Hecke ring."

An easy general result about endomorphism algebras of permutation modules shows that H_q has a basis $\{[\mathcal{O}]\}$ indexed by the set of orbits $\mathcal{O}$ for the diagonal

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action of G_q on $X_q \times X_q$, where $[\mathcal{O}]$ denotes the endomorphism

$$[\mathcal{O}](y) = \sum_{(x,y)\in\mathcal{O}} x$$

for any $y \in X_q$. The multiplication satisfies

$$[\mathcal{O}'][\mathcal{O}''] = \sum_{\mathcal{O}} m^{\mathcal{O}}_{\mathcal{O}',\mathcal{O}''}(q)[\mathcal{O}],$$

where $m^{\mathcal{O}}_{\mathcal{O}',\mathcal{O}''}(q) := \#\{y \in X_q \mid (x,y) \in \mathcal{O}', (y,z) \in \mathcal{O}''\}$ for any point $(x,z) \in \mathcal{O}$. For this to be useful, we need to understand the G_q-orbits on $X_q \times X_q$, which is equivalent to describing the B_q-orbits on X_q, or the (B_q, B_q)-double cosets in G_q. This follows from the *Bruhat decomposition*, which asserts that such double cosets are indexed canonically by the Weyl group W. We conclude that the G_q-orbits on $X_q \times X_q$ are parametrized by the Weyl group W, so for each $w \in W$ there is a unique orbit $\mathcal{O}_w \subseteq X_q \times X_q$ containing the point (wB_q, B_q). Thus, our basis for H_q is $\{[\mathcal{O}_w] \mid w \in W\}$, the key point being that it is indexed now by the set W which does not depend on the particular choice of q.

Iwahori's work implies that there are unique polynomials $m^w_{w',w''}(t)$ in an indeterminate t which evaluate to the above structure constants $m^{\mathcal{O}}_{\mathcal{O}',\mathcal{O}''}(q)$ for every prime power q and $\mathcal{O} = \mathcal{O}_w, \mathcal{O}' = \mathcal{O}_{w'}, \mathcal{O}'' = \mathcal{O}_{w''}$. This means it is possible to define a generic version of the Iwahori-Hecke algebra over the polynomial ring $\mathbb{Z}[t]$ from which all the algebras H_q above can be recovered by specializing at $t = q$. This generic algebra can also be defined by generators and relations; at the same time as doing this, we are going to renormalize the classical definition slightly, so let $v = \sqrt{t}$ be another indeterminate. Then the *generic Iwahori-Hecke algebra* H is the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $T_1, \dots, T_n$ subject to the relations

$$T_i^2 = (v - v^{-1})T_i + 1, \qquad \underbrace{T_iT_jT_i\cdots}_{m_{ij}\text{ times}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\text{ times}}. \tag{2}$$

The braid relations imply that there is a well-defined element $T_w \in H$ for every $w \in W$, such that $T_w = T_{i_1} \cdots T_{i_m}$ whenever $w = s_{i_1} \cdots s_{i_m}$ is a reduced expression; the elements $\{T_w \mid w \in W\}$ form a basis for H as a free $\mathbb{Z}[v, v^{-1}]$-module. The point then is that the specialization of H at $v = \sqrt{q}$ is isomorphic to the algebra H_q for any prime power q, i.e. there is an isomorphism

$$H \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \xrightarrow{\sim} H_q, \qquad v^{\ell(w)}T_w \otimes 1 \mapsto [\mathcal{O}_w],$$

where $\mathbb{C}$ is viewed as a $\mathbb{Z}[v, v^{-1}]$-module via $v \mapsto \sqrt{q}$. Note also by comparing the relations (1) and (2) that the specialization of H at $v = 1$ is isomorphic to the group algebra $\mathbb{C}W$ of the underlying Weyl group.

In their seminal 1979 paper, Kazhdan and Lusztig introduced another basis $\{C_w \mid w \in W\}$ for H. To define it, we need the *bar involution* $- : H \to H$, which is the unique ring automorphism such that $\overline{v} = v^{-1}$ and $\overline{T_i} = T_i^{-1}$. Observe for any $w \in W$ that $\overline{T_w} = T_{w^{-1}}^{-1} = T_w + (*)$ where $(*)$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of $T_{w'}$'s that are lower in the sense that $\ell(w') < \ell(w)$. More precisely, the $T_{w'}$'s that appear with non-zero coefficients in the expansion of $\overline{T_w}$ all satisfy $w' \leq w$, where $\leq$ is the *Bruhat order* on W. Using this observation and a little inductive argument, it follows for each $w \in W$ that there is a unique element $C_w \in H$ such that $\overline{C_w} = C_w$ and $C_w \equiv T_w \pmod{H_{<0}}$, where $H_{<0}$ is the $v^{-1}\mathbb{Z}[v^{-1}]$-span of the basis elements

$\{T_{w'} \mid w' \in W\}$. The elements $\{C_w \mid w \in W\}$ give the *Kazhdan-Lusztig basis* of H. One can show moreover that there are polynomials $P_{w',w}(v^2) \in \mathbb{Z}[v^2]$ such that

$$C_w = \sum_{w' \in W} v^{\ell(w')-\ell(w)} P_{w',w}(v^2) T_{w'}.$$

These are the *Kazhdan-Lusztig polynomials* introduced originally in [7]. They satisfy $P_{w,w}(v^2) = 1$, and $P_{w',w}(v^2) = 0$ unless $w' \leq w$ in the Bruhat order.

Many of the deepest results about Kazhdan-Lusztig polynomials rely on a geometric interpretation explained in [8]. Let G be a complex reductive algebraic group with underlying Dynkin diagram Γ, and let B be a Borel subgroup of G. The Weyl group W can be identified with $N_G(T)/T$, where $T < B$ is a maximal torus. For each $w \in W$, we consider the *Schubert varieties* $X(w) := \overline{BwB/B}$ in the flag variety $X := G/B$. Let IC_w denote the Goresky-Macpherson middle extension of the constant sheaf along the open embedding $j : BwB/B \hookrightarrow X(w)$, so that its cohomological shift $IC_w[\ell(w)]$ is a perverse sheaf. For $w' \leq w$, let $\mathcal{H}^i(IC_w)_{w'}$ denote the stalk of the ith cohomology sheaf of IC_w at the point $w'B \in X(w)$. Kazhdan and Lusztig showed that

$$P_{w',w}(v^2) = \sum_i \dim \mathcal{H}^i(IC_w)_{w'} v^i.$$

In particular, it follows that all the coefficients of $P_{w',w}(v^2)$ are non-negative integers; there is still no purely combinatorial proof of this positivity property.

The most celebrated result obtained using the above geometric interpretation is the *Kazhdan-Lusztig conjecture* proved by Beilinson and Bernstein [2] and Brylinski and Kashiwara [3]. This conjecture, which was formulated already in [7], relates Kazhdan-Lusztig polynomials to representation theory of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{t}$ be the Lie algebra of the maximal torus T, so that the Weyl group W acts naturally on $\mathfrak{t}^*$. Let $\rho \in \mathfrak{t}^*$ be the weight that is half the sum of the positive roots defined by the choice of B. For each $w \in W$, let $L(w)$ denote the irreducible highest weight module of highest weight $-w\rho - \rho$. The module $L(w)$ is the unique irreducible quotient of the so-called *Verma module* $M(w)$, the universal highest weight module of this highest weight. The Kazhdan-Lusztig conjecture gives the following explicit formula for the character of $L(w)$ in terms of the easily-computed characters of Verma modules:

$$\operatorname{ch} L(w) = \sum_{w' \leq w} (-1)^{\ell(w)-\ell(w')} P_{w',w}(1) \operatorname{ch} M(w').$$

This theorem is also a key ingredient in the completion of the proof of another major result from the early 1980s, the classification of primitive ideals in the universal enveloping algebra $U(\mathfrak{g})$. At the time of [7], fundamental work of Duflo, Jantzen, Joseph, Barbasch, Vogan and others had essentially reduced the classification of primitive ideals to the question of determining exactly when the irreducible modules $L(w)$ and $L(w')$ have the same annihilators in $U(\mathfrak{g})$, for $w, w' \in W$.

Kazhdan and Lusztig explained how to define *left cells* in Weyl groups in terms of certain leading coefficients of Kazhdan-Lusztig polynomials, and observed (modulo the proof of the Kazhdan-Lusztig conjecture) that $L(w)$ and $L(w')$ have the same annihilators in $U(\mathfrak{g})$ if and only if w and w' belong to the same left cell. To formulate the definition of left cell precisely, write $w \leftarrow_L w'$ if there exists a simple reflection s such that $C_s C_{w'}$ has non-zero C_w-coefficient when expanded in terms of the Kazhdan-Lusztig basis. Extend this to a pre-order $\leq_L$ on W, so $w \leq_L w'$ if there exists $w = w_0, \ldots, w_m = w'$ such that $w_{i-1} \leftarrow_L w_i$ for each $i = 1, \ldots, m$. Finally let $\sim_L$ be the smallest equivalence relation on W such that $\leq_L$ induces a well-defined partial order on the quotient $W/\sim$, and write $w <_L w'$ if $w \leq_L w'$ but $w \not\sim_L w'$. The left cells in W are then the $\sim_L$-equivalence classes. Given any left cell γ, there is a corresponding *left cell module*

$$S(\gamma) := \bigoplus_{w' \leq_L w} \mathbb{Z}[v, v^{-1}]C_{w'} \Big/ \bigoplus_{w' <_L w} \mathbb{Z}[v, v^{-1}]C_{w'},$$

where w is any fixed element of γ. It is easy to see from the above definitions that $S(\gamma)$ is a quotient of two left ideals in H, so that it is a left H-module.

In the most well-behaved case $\Gamma = A_n$, the combinatorics of left cells and left cell modules can be understood explicitly in terms of a certain map P from $W = S_{n+1}$ to the set of *standard tableaux* of size $(n + 1)$, i.e. Young diagrams filled with the integers $1, \ldots, (n+1)$ so that they are strictly increasing along rows and down columns. This map is half of the classical *Robinson-Schensted correspondence.* Very briefly, given $w \in W$, the tableau $P(w)$ is computed starting from the empty tableau by successively inserting $w(1), w(2), \ldots, w(n + 1)$ into the first row using the rule that a smaller entry bumps the next biggest entry in a row into the next row down, so that after each insertion one still has a standard tableau. For example, if w is the permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

then we need to insert the numbers $4, 3, 1, 5, 2$ in order, so 3 bumps 4 down then 1 bumps 3 down which bumps 4 further down, and finally 2 bumps 5 down, to obtain

$$P(w) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1} \end{array}$$

It is then an important theorem that $w \sim_L w'$ if and only if $P(w^{-1}) = P((w')^{-1})$. Moreover, given a left cell γ, the restriction of the map P defines a bijection between the elements of γ and the set of all standard λ-tableaux, where λ is the partition of $(n + 1)$ that gives the shape of the tableau $P(w^{-1})$ for $w \in \gamma$. Kazhdan and Lusztig observed that the left cell module $S(\gamma)$ actually depends up to canonical isomorphism only on the partition λ, so we can denote $S(\gamma)$ instead by $S(\lambda)$. It has a basis parametrized by standard λ-tableaux, arising from the images of $\{C_w \mid w \in \gamma\}$. Moreover the specialization of $S(\lambda)$ to a $\mathbb{C}W$-module is actually *irreducible* (something which is false for other Γ), and all irreducible $\mathbb{C}W$-modules arise in this way. Thus, in type A_n, the Kazhdan-Lusztig theory also produces a special basis for each irreducible $\mathbb{C}W$-module.

We have limited our discussion above just to the case of Weyl groups, but most of the definitions – though not the geometric and representation theoretic interpretations – make sense more generally for Hecke algebras associated to arbitrary Coxeter groups; see [12] for a detailed account and many open problems. The case of affine Weyl groups is particularly rich, as there is again a geometric interpretation, and Lusztig's work has revealed an intimate relationship between cells in affine Weyl groups and the geometry of the nilpotent cone. There is also a variation on Kazhdan-Lusztig polynomials adapted to the representation theory of real Lie groups, called *Kazhdan-Lusztig-Vogan polynomials*; it is these polynomials which were recently computed for E_8 in a computational project initiated by Vogan.

Quantized enveloping algebras, Ringel-Hall algebras and canonical bases. There is a close analogy between the story just told and some more recent developments in the theory of quantum groups. For simplicity, let us assume from now on that the Dynkin diagram Γ is *simply-laced*, so type A_n, D_n or E_n only. Fix also an orientation on the edges of Γ, so that the edges become arrows, and the graph Γ becomes a rather special sort of quiver Q.

Let F be a fixed ground field. A *representation* V of the quiver Q means an assignment of finite dimensional F-vector spaces $V_1, \dots, V_n$ to the vertices and linear maps $f_{i\to j} : V_i \to V_j$ to the arrows. Given another such representation V', a morphism $\theta : V \to V'$ is a collection of linear maps $\theta_i : V_i \to V_i'$ such that $\theta_j \circ f_{i\to j} = f'_{i\to j} \circ \theta_i$ for every arrow. This defines an abelian category $\mathrm{Rep}(Q)$.

Of course the category $\mathrm{Rep}(Q)$ can be defined for an arbitrary quiver, not just one arising from a simply-laced Dynkin diagram. By a classic theorem due to Gabriel [4], the quivers coming from Dynkin diagrams are exactly the ones for which $\mathrm{Rep}(Q)$ is of *finite representation type*, that is, there are only finitely many isomorphism classes of indecomposable objects. Moreover, if we let $\mathfrak{g}$ be the finite dimensional simple Lie algebra over $\mathbb{C}$ of type Γ and fix a choice $\alpha_1, \dots, \alpha_n$ of simple roots in the corresponding root system, then the map

$$V \mapsto \sum_i (\dim V_i)\alpha_i$$

gives a bijection between the isomorphism classes of indecomposable objects in $\mathrm{Rep}(Q)$ and the set R^+ of positive roots in the root system of $\mathfrak{g}$. From this, we get a bijection from the set of isomorphism classes of arbitrary objects $V \in \mathrm{Rep}(Q)$ to the set of functions $\lambda : R^+ \to \mathbb{Z}_{\geq 0}$, such that $[V] \mapsto \lambda$ if the indecomposable module parametrized by $\alpha \in R^+$ appears as a summand of V with multiplicity $\lambda(\alpha)$; we say simply that V is of *type* λ if this is the case.

Gabriel's theorem is the first of long line of remarkable results relating representations of quivers to the structure of Kac-Moody Lie algebras. We are going to focus on just one of these, namely, Ringel's Hall algebra construction of the positive part of the quantized enveloping algebra of $\mathfrak{g}$. Suppose for this that the ground field F used to define the category $\mathrm{Rep}(Q)$ is the finite field $\mathbb{F}_q$ for some q. The *Hall algebra* $\mathscr{H}_q$ of the category $\mathrm{Rep}(Q)$ is the free $\mathbb{Z}$-algebra with basis $\{[V]\}$ indexed by the isomorphism classes of objects in $\mathrm{Rep}(Q)$, and multiplication defined by the formula

$$[V'][V''] = \sum_{[V]} h^V_{V',V''}(q)[V]$$

where $h^V_{V',V''}(q) := \#\{\text{subobjects } U \leq V \text{ such that } V/U \cong V' \text{ and } U \cong V''\}$. It is an easy exercise to see that $\mathscr{H}_q$ is associative and unital. This definition is a special case of a very general notion of Hall algebra originating in classical works of Steinitz and Hall, who considered this algebra for the Kronecker quiver.

Ringel showed for any $\lambda, \lambda', \lambda'' : R^+ \to \mathbb{Z}_{\geq 0}$ that there exist polynomials $h^\lambda_{\lambda',\lambda''}(t)$ in an indeterminate t that evaluate to the structure constants $h^V_{V',V''}(q)$ in the algebra $\mathscr{H}_q$ for every prime power q and $V, V', V'' \in \mathrm{Rep}(Q)$ of types $\lambda, \lambda', \lambda''$, respectively. These are Ringel's *Hall polynomials*. Using them, it is possible to define a generic version of the Hall algebra. Actually, we skip this and pass directly to the definition of the *Ringel-Hall algebra* $\mathscr{H}$, in which there is an additional twist in the multiplication. By definition, $\mathscr{H}$ is the free $\mathbb{Z}[v, v^{-1}]$-algebra with basis $\{E_\lambda \mid \lambda : R^+ \to \mathbb{Z}_{\geq 0}\}$ and multiplication defined by the rule

$$E_{\lambda'} E_{\lambda''} = v^{-\langle \lambda'', \lambda' \rangle} \sum_{\lambda} h^{\lambda}_{\lambda', \lambda''}(v^2) E_\lambda,$$

where $\langle \lambda'', \lambda' \rangle$ denotes the integer $\dim \mathrm{Hom}(V'', V') - \dim \mathrm{Ext}^1(V'', V')$ for $V', V'' \in \mathrm{Rep}(Q)$ of types λ', λ'', respectively. It is important to note here that this integer is itself independent of q, which is established by showing that

$$\langle \lambda'', \lambda' \rangle = \sum_i (\dim V_i'')(\dim V_i') - \sum_{i \to j} (\dim V_i'')(\dim V_j').$$

In [13], Ringel uncovered a remarkable connection between the algebra $\mathscr{H}$ and the quantized enveloping algebra of the Lie algebra $\mathfrak{g}$, which was introduced already by Drinfeld and Jimbo in the mid-1980s. We just need the positive part of this algebra, which is the $\mathbb{Q}(v)$-algebra U^+ on generators $E_1, \dots, E_n$ subject only to the relations

$$\begin{array}{rl} E_i^2 E_j + E_j E_i^2 = (v + v^{-1}) E_i E_j E_i & \text{if } i \text{ and } j \text{ are connected by an edge in } \Gamma, \\ E_i E_j = E_j E_i & \text{otherwise.} \end{array}$$

Ringel's work implies that there is an *isomorphism* between $\mathscr{H}$ and the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U^+ generated by the *divided powers* $E_i^{(r)} := E_i^r/[r]!$ for all i and $r \geq 1$, where $[r]! := \prod_{s=1}^r \frac{v^s - v^{-s}}{v - v^{-1}}$. The isomorphism is uniquely determined by the property that it maps E_λ to $E_i^{(r)}$ if $\lambda : R^+ \to \mathbb{Z}_{\geq 0}$ is the function taking value r on α_i and zero on all other positive roots. Henceforth, we will identify $\mathscr{H}$ with a subalgebra of U^+ via this map. In particular, the defining basis $\{E_\lambda | \lambda : R^+ \to \mathbb{Z}_{\geq 0}\}$ for $\mathscr{H}$ becomes a basis for U^+. While the algebra U^+ itself depends only on the underlying Dynkin diagram Γ, not on the choice of Q, this basis for U^+ definitely does depend on Q. The bases arising in this way are special instances of the *PBW-type bases* constructed subsequently by Lusztig via a braid group action; see [11].

Ringel's theorem was the starting point for Lusztig's definition in [10] of the *canonical basis* of the algebra U^+. Introduce the *bar involution* $- : U^+ \to U^+$, the unique $\mathbb{Q}$-algebra automorphism such that $\overline{v} = v^{-1}$ and $\overline{E_i} = E_i$. Lusztig showed that there was a certain partial order $\leq$ on the set of all functions $\lambda : R^+ \to \mathbb{Z}_{\geq 0}$ such that $\overline{E_\lambda} = E_\lambda + (*)$ where $(*)$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of E_μ's for $\mu < \lambda$. The original proof of this statement explained in [10, §7] is purely algebraic in nature, but depends essentially on facts about the representation theory of Q, in particular, the partial order $\leq$ is related to the underlying Auslander-Reiten quiver. Using this triangularity result and making the same inductive argument as in the definition of the Kazhdan-Lusztig basis for Iwahori-Hecke algebras, it follows that there is a unique element $B_\lambda \in U^+$ such that $\overline{B_\lambda} = B_\lambda$ and $B_\lambda \equiv E_\lambda \pmod{U^+_{<0}}$, where $U^+_{<0}$ is the $v^{-1}\mathbb{Z}[v^{-1}]$-span of the PBW-type basis. In this way, we obtain the *canonical basis* $\{B_\lambda \mid \lambda : R^+ \to \mathbb{Z}_{\geq 0}\}$ for U^+. Unlike the PBW-type basis, this basis depends only on Γ, not on the quiver Q. For example, for the (deceptively simple) case $\Gamma = A_2$, the canonical basis consists of the elements

$$\{E_1^{(b)} E_2^{(b+c)} E_1^{(a)} \mid c \geq a\} \cup \{E_2^{(c)} E_1^{(a+b)} E_2^{(b)} \mid c < a\}.$$

In the second half of the paper [10], Lusztig went on to explain a geometric interpretation of the canonical basis using the theory of perverse sheaves, which produces some remarkable positivity properties. Subsequently Lusztig extended his geometric construction to symmetrizable Kac-Moody Lie algebras; see [11].

Another key feature of the canonical basis of U^+ is that it induces canonical bases in every finite dimensional irreducible representation of $\mathfrak{g}$. More precisely, if we fix a lowest weight vector v_- in such a representation, the non-zero vectors in the set $\{B_\lambda v_- \mid \lambda : R^+ \to \mathbb{Z}_{\geq 0}\}$ give a basis for the representation. As in the case of Iwahori-Hecke algebras, it again makes sense to specialize at $v = 1$. On doing this one obtains a canonical basis for the positive part of $U(\mathfrak{g})$ and for all the finite dimensional irreducible representations of the underlying semisimple Lie algebra $\mathfrak{g}$.

Independently and at the same time as Lusztig's work, Kashiwara introduced the theory of *crystal bases* for quantized enveloping algebras and their integrable representations; see [6]. In a precise sense, these are bases at $v = \infty$. This uncovered a tremendously rich structure which has revolutionized the combinatorial representation theory of semisimple Lie algebras. Furthermore, Kashiwara explained how to lift crystal bases to obtain a *global crystal basis* for U^+ itself, which was subsequently shown by Grojnowski and Lusztig to coincide with Lusztig's canonical basis. We stress that Kashiwara's approach does not rely on any geometry or representation theory of quivers.

To round off this brief survey, we wish to mention one much more recent breakthrough which provides an elementary representation theoretic interpretation of the canonical basis. This involves some remarkable new algebras discovered independently by Khovanov and Lauda [9] and Rouquier [14], which may be called *quiver Hecke algebras*. These algebras resemble affine Hecke algebras in some respects, and carry a natural $\mathbb{Z}$-grading. Khovanov and Lauda formulated a precise conjecture relating the representation theory of quiver Hecke algebras to the canonical basis of U^+. Roughly, the canonical basis plays the same role in the Khovanov-Lauda conjecture for quiver Hecke algebras as the Kazhdan-Lusztig basis plays in the Kazhdan-Lusztig conjecture for semisimple Lie algebras. For simply-laced types, the Khovanov-Lauda conjecture has already been proved by Varagnolo and Vasserot [15], exploiting Lusztig's geometric approach to the canonical basis.

The book. Like this review, the book by Deng, Du, Parshall and Wang covers a tremendous amount of ground. It is concerned primarily with the algebraic and combinatorial aspects of the theories mentioned above. It does not attempt to treat the more geometric aspects, nor is it concerned with the striking applications to representation theory such as the Kazhdan-Lusztig conjectures; these matters are mentioned only very briefly in notes at the end of various chapters.

It covers in detail the Kazhdan-Lusztig theory for Iwahori-Hecke algebras, with special emphasis on the combinatorics of cells and cell modules in type A_n. It also covers in detail the connection between the Ringel-Hall algebra of a quiver and the positive part of the corresponding quantized enveloping algebra, both in finite type, and also affine type where a theorem of Green becomes essential. A particular strength of the book is the careful treatment of quivers with automorphisms, which are needed to extend the constructions mentioned above in simply-laced types to the twisted types B_n, C_n, F_4 and G_2. It also includes a great deal of background from the representation theory of finite dimensional algebras, including the necessary results from Auslander-Reiten theory. This culminates in the construction, essentially following Lusztig's first approach from [10, §7], of the canonical basis. Lusztig's paper [10] only considers the simply-laced types, and his subsequent work uses only the geometric approach, so substantial parts of this development in the twisted cases cannot be found elsewhere in the literature.

A third major part of the book is concerned with the bridge between the Kazhdan-Lusztig and canonical bases in type A_n. This is based in part on the article of Beilinson, Lusztig and Macpherson [1], which gives a geometric realization of the *quantum Schur algebra* S_q. This algebra appeared previously in works of Jimbo and Dipper-James, and can be constructed as the endomorphism algebra of a permutation module arising from the natural action of $GL_{n+1}(\mathbb{F}_q)$ on certain *partial* flag varieties. The definition of the Kazhdan-Lusztig basis and the associated theory of left cells extends very naturally from the Iwahori-Hecke algebra to the quantum Schur algebra; at the level of combinatorics, the Robinson-Schensted correspondence gets replaced by the more general Robinson-Schensted-Knuth correspondence. From there, it is possible to lift the Kazhdan-Lusztig basis for the quantum Schur algebra to obtain a basis for a modified version of the entire quantized enveloping algebra in type A_n, part of which is known to recover the canonical basis. Thus in type A_n, the theories of Kazhdan-Lusztig bases and canonical bases are intimately related.

This book is a welcome addition to the literature in this subject. The exposition is very clear and comprehensive, and does not assume much beyond a basic knowledge of semisimple Lie algebras. The book should be of particular value to graduate students seeking to acquire the broad background knowledge needed to become successful researchers in this vibrant area.

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