PRINCIPAL W-ALGEBRAS FOR GL$(m|n)$

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Abstract. We consider the (finite) W-algebra $W_{m|n}$ attached to the principal nilpotent orbit in the general linear Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. Our main result gives an explicit description of $W_{m|n}$ as a certain truncation of a shifted version of the Yangian $Y(\mathfrak{gl}_{1|1})$. We also show that $W_{m|n}$ admits a triangular decomposition and construct its irreducible representations.

1. Introduction

A *(finite) W-algebra* is a certain filtered deformation of the Slodowy slice to a nilpotent orbit in a complex semisimple Lie algebra $\mathfrak{g}$. Although the terminology is more recent, the construction has its origins in the classic work of Kostant [K]. In particular Kostant showed that the principal W-algebra—the one associated to the principal nilpotent orbit in $\mathfrak{g}$—is isomorphic to the center of the universal enveloping algebra $U(\mathfrak{g})$. In the last few years there has been some substantial progress in understanding W-algebras for other nilpotent orbits, thanks to works of Premet, Losev and others; see [L] for a survey. The story is most complete (also easiest) for $\mathfrak{sl}_n(\mathbb{C})$. In this case the W-algebras are closely related to *shifted Yangians*; see [BK1].

Analogues of W-algebras have also been defined for Lie superalgebras; see for example the work of De Sole and Kac [DK, §5.2] (where they are defined in terms of BRST cohomology) or the more recent paper of Zhao [Z] (which focusses mainly on the queer Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$). In this article we consider the easiest of all the "super" situations: the *principal W-algebra $W_{m|n}$ for the general linear Lie superalgebra* $\mathfrak{gl}_{m|n}(\mathbb{C})$. Our main result gives an explicit isomorphism between $W_{m|n}$ and a certain truncation of a shifted subalgebra of the Yangian $Y(\mathfrak{gl}_{1|1})$; see Theorem 4.5. Its proof is very similar to the proof of the analogous result for nilpotent matrices of Jordan type (m,n) in $\mathfrak{gl}_{m+n}(\mathbb{C})$ from [BK1].

The (super)algebra $W_{m|n}$ turns out to be quite close to being supercommutative. More precisely, we show that it admits a triangular decomposition

$$W_{m|n} = W^-_{m|n} W^0_{m|n} W^+_{m|n}$$

in which $W^-_{m|n}$ and $W^+_{m|n}$ are exterior algebras of dimension $2^{\min(m,n)}$ and $W^0_{m|n}$ is a symmetric algebra of rank $(m+n)$; see Theorem 6.1. This implies that all the irreducible $W_{m|n}$-modules are finite dimensional; see Theorem 7.2. We show further that they all arise as certain tensor products of irreducible $\mathfrak{gl}_{1|1}(\mathbb{C})$- and $\mathfrak{gl}_1(\mathbb{C})$-modules; see Theorem 8.4. In particular, all irreducible $W_{m|n}$-modules are

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of dimension dividing $2^{\min(m,n)}$. A closely related assertion is that all irreducible highest weight representations of $Y(\mathfrak{gl}_{1|1})$ are tensor products of evaluation modules; this is similar to a well-known phenomenon for $Y(\mathfrak{gl}_2)$ going back to [T].

Some related results about $W_{m|n}$ have been obtained independently by Poletaeva and Serganova [PS]. In fact, the connection between $W_{m|n}$ and the Yangian $Y(\mathfrak{gl}_{1|1})$ was foreseen long ago by Briot and Ragoucy [BR]. Briot and Ragoucy also looked at certain non-principal nilpotent orbits which they assert are connected to higher rank super Yangians, although we do not understand their approach. It should be possible to combine the methods of this article with those of [BK1] to establish such a connection for *all* nilpotent orbits in $\mathfrak{gl}_{m|n}(\mathbb{C})$. However this is not trivial and will require some new presentations for the higher rank super Yangians adapted to arbitrary parity sequences; the ones in [G, P] are not sufficient as they only apply to the standard parity sequence.

By analogy with Kostant's results from [K] our expectation is that $W_{m|n}$ will play a distinguished role in the representation theory of $\mathfrak{gl}_{m|n}(\mathbb{C})$. In a forthcoming article [BBG], we will investigate the *Whittaker coinvariants functor* H_0, a certain exact functor from the analogue of category $\mathcal{O}$ for $\mathfrak{gl}_{m|n}(\mathbb{C})$ to the category of finite dimensional $W_{m|n}$-modules. We view this as a replacement for Soergel's functor $\mathbb{V}$ from [S]; see also [B]. We will show that H_0 sends irreducible modules in $\mathcal{O}$ to irreducible $W_{m|n}$-modules or zero, and that all irreducible $W_{m|n}$-modules occur in this way; this should be compared with the analogous result for parabolic category $\mathcal{O}$ for $\mathfrak{gl}_{m+n}(\mathbb{C})$ obtained in [BK2, Theorem E]. We will also use properties of H_0 to prove that the center of $W_{m|n}$ is isomorphic to the center of the universal enveloping superalgebra of $\mathfrak{gl}_{m|n}(\mathbb{C})$.

Notation. We denote the parity of a homogeneous vector x in a $\mathbb{Z}/2$-graded vector space by $|x| \in \{\bar{0}, \bar{1}\}$. A *superalgebra* means a $\mathbb{Z}/2$-graded algebra over $\mathbb{C}$. For homogeneous x and y in an associative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, their *supercommutator* is $[x, y] := xy - (-1)^{|x||y|}yx$. We say that A is *supercommutative* if $[x, y] = 0$ for all homogeneous $x, y \in A$. Also for homogeneous $x_1, \dots, x_n \in A$, an *ordered supermonomial* in $x_1, \dots, x_n$ means a monomial of the form $x_1^{i_1} \cdots x_n^{i_n}$ for $i_1, \dots, i_n \geq 0$ such that $i_j \leq 1$ if x_j is odd.

2. Shifted Yangians

Recall that $\mathfrak{gl}_{m|n}(\mathbb{C})$ is the Lie superalgebra of all $(m+n) \times (m+n)$ complex matrices under the supercommutator, with $\mathbb{Z}/2$-grading defined so that the matrix unit $e_{i,j}$ is even if $1 \leq i, j \leq m$ or $m+1 \leq i, j \leq m+n$, and $e_{i,j}$ is odd otherwise. We denote its universal enveloping superalgebra by $U(\mathfrak{gl}_{m|n})$; it has basis given by all ordered supermonomials in the matrix units.

The Yangian $Y(\mathfrak{gl}_{m|n})$ was introduced originally by Nazarov [N] ; see also [G]. We only need here the special case of $Y = Y(\mathfrak{gl}_{1|1})$. For its definition we fix a choice of *parity sequence*

$$(|1|, |2|) \in \mathbb{Z}/2 \times \mathbb{Z}/2 \tag{2.1}$$

with $|1| \neq |2|$. All subsequent notation in the remainder of the article depends implicitly on this choice. Then we define Y to be the associative superalgebra on generators $\{t_{i,j}^{(r)} \mid 1 \leq i, j \leq 2, r > 0\}$, with $t_{i,j}^{(r)}$ of parity $|i| + |j|$, subject to the relations

$$\left[t_{i,j}^{(r)}, t_{p,q}^{(s)}\right] = (-1)^{|i||j|+|i||p|+|j||p|} \sum_{a=0}^{\min(r,s)-1} \left(t_{p,j}^{(a)} t_{i,q}^{(r+s-1-a)} - t_{p,j}^{(r+s-1-a)} t_{i,q}^{(a)}\right),$$

adopting the convention that $t_{i,j}^{(0)} = \delta_{i,j}$ (Kronecker delta).

Remark 2.1. In the literature, one typically only finds results about $Y(\mathfrak{gl}_{1|1})$ proved for the definition coming from the parity sequence $(|1|, |2|) = (\bar{0}, \bar{1})$. To aid in translating between this and the other possibility, we note that the map $t_{i,j}^{(r)} \mapsto (-1)^r t_{i,j}^{(r)}$ defines an isomorphism between the realizations of $Y(\mathfrak{gl}_{1|1})$ arising from the two choices of parity sequence.

As in [N], we introduce the generating function

$$t_{i,j}(u) := \sum_{r\geq 0} t_{i,j}^{(r)} u^{-r} \in Y[\![u^{-1}]\!].$$

Then Y is a Hopf superalgebra with comultiplication Δ and counit ε given in terms of generating functions by

$$\Delta(t_{i,j}(u)) = \sum_{h=1}^{2} t_{i,h}(u) \otimes t_{h,j}(u), \tag{2.2}$$

$$\varepsilon(t_{i,j}(u)) = \delta_{i,j}. \tag{2.3}$$

There are also algebra homomorphisms

$$\mathrm{in} : U(\mathfrak{gl}_{1|1}) \to Y, \qquad e_{i,j} \mapsto (-1)^{|i|} t_{i,j}^{(1)}, \tag{2.4}$$

$$\mathrm{ev} : Y \to U(\mathfrak{gl}_{1|1}), \qquad t_{i,j}^{(r)} \mapsto \delta_{r,0}\delta_{i,j} + (-1)^{|i|}\delta_{r,1} e_{i,j}. \tag{2.5}$$

The composite ev $\circ$ in is the identity, hence in is injective and ev is surjective. We call ev the *evaluation homomorphism*.

We need another set of generators for Y called *Drinfeld generators*. To define these, we consider the Gauss factorization $T(u) = F(u)D(u)E(u)$ of the matrix

$$T(u) := \begin{pmatrix} t_{1,1}(u) & t_{1,2}(u) \\ t_{2,1}(u) & t_{2,2}(u) \end{pmatrix}.$$

This defines power series $d_i(u), e(u), f(u) \in Y[\![u^{-1}]\!]$ such that

$$D(u) = \begin{pmatrix} d_1(u) & 0 \\ 0 & d_2(u) \end{pmatrix}, \quad E(u) = \begin{pmatrix} 1 & e(u) \\ 0 & 1 \end{pmatrix}, \quad F(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix}.$$

Thus we have that

$$d_1(u) = t_{1,1}(u), \qquad d_2(u) = t_{2,2}(u) - t_{2,1}(u)t_{1,1}(u)^{-1}t_{1,2}(u), \tag{2.6}$$

$$e(u) = t_{1,1}(u)^{-1}t_{1,2}(u), \qquad f(u) = t_{2,1}(u)t_{1,1}(u)^{-1}. \tag{2.7}$$

Equivalently,

$$t_{1,1}(u) = d_1(u), \qquad t_{2,2}(u) = d_2(u) + f(u)d_1(u)e(u), \tag{2.8}$$

$$t_{1,2}(u) = d_1(u)e(u), \qquad t_{2,1}(u) = f(u)d_1(u). \tag{2.9}$$

The Drinfeld generators are the elements $d_i^{(r)}, e^{(r)}$ and $f^{(r)}$ of Y defined from the expansions $d_i(u) = \sum_{r\geq 0} d_i^{(r)} u^{-r}$, $e(u) = \sum_{r\geq 1} e^{(r)}u^{-r}$ and $f(u) = \sum_{r\geq 1} f^{(r)}u^{-r}$. Also define $\tilde{d}_i^{(r)} \in Y$ from the identity $\tilde{d}_i(u) = \sum_{r\geq 0} \tilde{d}_i^{(r)} u^{-r} := d_i(u)^{-1}$.

Theorem 2.2. [G, Theorem 3] *The superalgebra Y is generated by the even elements $\{d_i^{(r)} \mid i = 1, 2, r > 0\}$ and the odd elements $\{e^{(r)}, f^{(r)} \mid r > 0\}$ subject only to the following relations:*

$$[d_i^{(r)}, d_j^{(s)}] = 0, \qquad [e^{(r)}, f^{(s)}] = (-1)^{|1|} \sum_{a=0}^{r+s-1} \tilde{d}_1^{(a)} d_2^{(r+s-1-a)},$$

$$[e^{(r)}, e^{(s)}] = 0, \qquad [d_i^{(r)}, e^{(s)}] = (-1)^{|1|} \sum_{a=0}^{r-1} d_i^{(a)} e^{(r+s-1-a)},$$

$$[f^{(r)}, f^{(s)}] = 0, \qquad [d_i^{(r)}, f^{(s)}] = -(-1)^{|1|} \sum_{a=0}^{r-1} f^{(r+s-1-a)} d_i^{(a)}.$$

Here $d_i^{(0)} = 1$ and $\tilde{d}_i^{(r)}$ is defined recursively from $\sum_{a=0}^{r} \tilde{d}_i^{(a)} d_i^{(r-a)} = \delta_{r,0}$.

Remark 2.3. By [G, Theorem 4] the coefficients $\{c^{(r)} \mid r > 0\}$ of the power series

$$c(u) = \sum_{r\geq 0} c^{(r)} u^{-r} := \tilde{d}_1(u) d_2(u) \tag{2.10}$$

generate the center of Y. Moreover, $[e^{(r)}, f^{(s)}] = (-1)^{|1|} c^{(r+s-1)}$, so these supercommutators are central.

Remark 2.4. Using the relations in Theorem 2.2, one can check that Y admits an algebra automorphism

$$\zeta : Y \to Y, \qquad d_1^{(r)} \mapsto \tilde{d}_2^{(r)}, d_2^{(r)} \mapsto \tilde{d}_1^{(r)}, e^{(r)} \mapsto -f^{(r)}, f^{(r)} \mapsto -e^{(r)}. \tag{2.11}$$

By [G, Proposition 4.3], this satisfies

$$\Delta \circ \zeta = P \circ (\zeta \otimes \zeta) \circ \Delta \tag{2.12}$$

where $P(x \otimes y) = (-1)^{|x||y|} y \otimes x$.

Proposition 2.5. *The comultiplication Δ is given on Drinfeld generators by the following:*

$$\Delta(d_1(u)) = d_1(u) \otimes d_1(u) + d_1(u)e(u) \otimes f(u)d_1(u),$$

$$\Delta(\tilde{d}_1(u)) = \sum_{n\geq 0} (-1)^{\lceil n/2 \rceil} e(u)^n \tilde{d}_1(u) \otimes \tilde{d}_1(u) f(u)^n,$$

$$\Delta(d_2(u)) = \sum_{n\geq 0} (-1)^{\lfloor n/2 \rfloor} d_2(u) e(u)^n \otimes f(u)^n d_2(u),$$

$$\Delta(\tilde{d}_2(u)) = \tilde{d}_2(u) \otimes \tilde{d}_2(u) - e(u)\tilde{d}_2(u) \otimes \tilde{d}_2(u) f(u),$$

$$\Delta(e(u)) = 1 \otimes e(u) - \sum_{n\geq 1}(-1)^{\lceil n/2\rceil} e(u)^n \otimes \tilde{d}_1(u) f(u)^{n-1} d_2(u),$$
$$\Delta(f(u)) = f(u) \otimes 1 - \sum_{n\geq 1}(-1)^{\lceil n/2\rceil} d_2(u) e(u)^{n-1} \tilde{d}_1(u) \otimes f(u)^n.$$

Proof. Check the formulae for $d_1(u), \tilde{d}_1(u)$ and $e(u)$ directly using (2.2) and (2.6)–(2.7). The other formulae then follow using (2.12). $\square$

Here is the *PBW theorem* for Y.

Theorem 2.6 ([G, Theorem 1])**.** *Order the set $\{t_{i,j}^{(r)}\,|\,1 \leq i,j \leq 2, r > 0\}$ in some way. The ordered supermonomials in these generators give a basis for Y.*

There are two important filtrations on Y. First we have the *Kazhdan filtration* which is defined by declaring that the generator $t_{i,j}^{(r)}$ is in degree r, i.e. the filtered degree r part $F_r Y$ of Y with respect to the Kazhdan filtration is the span of all monomials of the form $t_{i_1,j_1}^{(r_1)} \cdots t_{i_n,j_n}^{(r_n)}$ such that $r_1 + \cdots + r_n \leq r$. The defining relations imply that the associated graded superalgebra $\operatorname{gr} Y$ is supercommutative. Let $\mathfrak{gl}_{1|1}[x]$ denote the current Lie superalgebra $\mathfrak{gl}_{1|1}(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[x]$ with basis $\{e_{i,j}x^r\,|\,1 \leq i,j \leq 2, r \geq 0\}$. Then Theorem 2.6 implies that $\operatorname{gr} Y$ can be identified with the symmetric superalgebra $S(\mathfrak{gl}_{1|1}[x])$ of the vector superspace $\mathfrak{gl}_{1|1}[x]$ so that $\operatorname{gr}_r t_{i,j}^{(r)} = (-1)^{|i|} e_{i,j} x^{r-1}$.

The other filtration on Y, which we call the *Lie filtration*, is defined similarly by declaring that $t_{i,j}^{(r)}$ is in degree $r-1$. In this case we denote the filtered degree r part of Y by $F'_r Y$ and the associated graded superalgebra by $\operatorname{gr}' Y$. By Theorem 2.6 and the defining relations once again, $\operatorname{gr}' Y$ can be identified with the universal enveloping superalgebra $U(\mathfrak{gl}_{1|1}[x])$ so that $\operatorname{gr}'_{r-1} t_{i,j}^{(r)} = (-1)^{|i|} e_{i,j} x^{r-1}$. The Drinfeld generators $d_i^{(r)}, e^{(r)}$ and $f^{(r)}$ all lie in $F'_{r-1} Y$ and we have that

$$\operatorname{gr}'_{r-1} d_i^{(r)} = \operatorname{gr}'_{r-1} t_{i,i}^{(r)}, \qquad \operatorname{gr}'_{r-1} e^{(r)} = \operatorname{gr}'_{r-1} t_{1,2}^{(r)}, \qquad \operatorname{gr}'_{r-1} f^{(r)} = \operatorname{gr}'_{r-1} t_{2,1}^{(r)}.$$

(The situation for the Kazhdan filtration is more complicated: although $d_i^{(r)}, e^{(r)}$ and $f^{(r)}$ do all lie in $F_r Y$, their images in $\operatorname{gr}_r Y$ are not in general equal to the images of $t_{i,i}^{(r)}, t_{1,2}^{(r)}$ or $t_{2,1}^{(r)}$, but they can expressed in terms of them via (2.6)–(2.7).)

Combining the preceding discussion of the Lie filtration with Theorem 2.6, we obtain the following basis for Y in terms of Drinfeld generators. (One can also deduce this by working with the Kazhdan filtration and using (2.6)–(2.9).)

Corollary 2.7. *Order the set $\{d_i^{(r)}\,|\,i = 1,2, r > 0\} \cup \{e^{(r)}, f^{(r)}\,|\,r > 0\}$ in some way. The ordered supermonomials in these generators give a basis for Y.*

Now we are ready to introduce the *shifted Yangians* for $\mathfrak{gl}_{1|1}(\mathbb{C})$. This parallels the definition of shifted Yangians in the purely even case from [BK1, §2]. Let $\sigma = (s_{i,j})_{1\leq i,j\leq 2}$ be a 2×2 matrix of non-negative integers with $s_{1,1} = s_{2,2} = 0$. We refer to such a matrix as a *shift matrix*. Let Y_σ be the superalgebra with even generators $\{d_i^{(r)}|i = 1,2, r > 0\}$ and odd generators $\{e^{(r)}|r > s_{1,2}\} \cup \{f^{(r)}|r > s_{2,1}\}$ subject to all of the relations from Theorem 2.2 that make sense, bearing in mind that we no longer have available the generators $e^{(r)}$ for $0 < r \leq s_{1,2}$ or $f^{(r)}$

for $0 < r \leq s_{2,1}$. Clearly there is a homomorphism $Y_\sigma \to Y$ which sends the generators of Y_σ to the generators with the same name in Y.

Theorem 2.8. *Order the set $\{d_i^{(r)} | i = 1, 2, r > 0\} \cup \{e^{(r)} | r > s_{1,2}\} \cup \{f^{(r)} | r > s_{2,1}\}$ in some way. The ordered supermonomials in these generators give a basis for Y_σ. In particular, the homomorphism $Y_\sigma \to Y$ is injective.*

Proof. It is easy to see from the defining relations that the monomials span, and their images in Y are linearly independent by Corollary 2.7. □

From now on we will identify Y_σ with a subalgebra of Y via the injective homomorphism $Y_\sigma \hookrightarrow Y$. The Kazhdan and Lie filtrations on Y induce filtrations on Y_σ such that $\operatorname{gr} Y_\sigma \subseteq \operatorname{gr} Y$ and $\operatorname{gr}' Y_\sigma \subseteq \operatorname{gr}' Y$. Let $\mathfrak{gl}_{1|1}^\sigma[x]$ be the Lie subalgebra of $\mathfrak{gl}_{1|1}[x]$ spanned by the vectors $e_{i,j} x^r$ for $1 \leq i, j \leq 2$ and $r \geq s_{i,j}$. Then we have that $\operatorname{gr} Y_\sigma = S(\mathfrak{gl}_{1|1}^\sigma[x])$ and $\operatorname{gr}' Y_\sigma = U(\mathfrak{gl}_{1|1}^\sigma[x])$.

Remark 2.9. Given another shift matrix $\sigma' = (s'_{i,j})_{1 \leq i,j \leq 2}$ with $s'_{2,1} + s'_{1,2} = s_{2,1} + s_{1,2}$ there is an isomorphism

$$\iota : Y_\sigma \xrightarrow{\sim} Y_{\sigma'}, \quad d_i^{(r)} \mapsto d_i^{(r)}, e^{(r)} \mapsto e^{(s'_{1,2} - s_{1,2} + r)}, f^{(r)} \mapsto f^{(s'_{2,1} - s_{2,1} + r)}. \tag{2.13}$$

This follows from the defining relations. Thus, up to isomorphism, Y_σ depends only on the integer $s_{2,1} + s_{1,2} \geq 0$, not on σ itself. Beware though that the isomorphism ι does not respect the Kazhdan or Lie filtrations.

For $\sigma \neq 0$, Y_σ is not a Hopf subalgebra of Y. However there are some useful comultiplication-like homomorphisms between different shifted Yangians. To start with, let σ^{up} (resp. σ^{lo}) be the upper (resp. lower) triangular shift matrix obtained from σ by setting $s_{2,1}$ (resp. $s_{1,2}$) equal to zero. Then, by Proposition 2.5, the restriction of the comultiplication Δ on Y gives a homomorphism

$$\Delta : Y_\sigma \to Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}}. \tag{2.14}$$

The remaining comultiplication-like homomorphisms involve the universal enveloping algebra $U(\mathfrak{gl}_1) = \mathbb{C}[e_{1,1}]$. Assuming that $s_{1,2} > 0$, let σ_+ be the shift matrix obtained from σ by subtracting 1 from the entry $s_{1,2}$. Then the relations imply that there is a well-defined algebra homomorphism

$$\Delta_+ : Y_\sigma \to Y_{\sigma_+} \otimes U(\mathfrak{gl}_1), \tag{2.15}$$

$$d_1^{(r)} \mapsto d_1^{(r)} \otimes 1, \qquad d_2^{(r)} \mapsto d_2^{(r)} \otimes 1 + (-1)^{|2|} d_2^{(r-1)} \otimes e_{1,1},$$

$$e^{(r)} \mapsto e^{(r)} \otimes 1 + (-1)^{|2|} e^{(r-1)} \otimes e_{1,1}, \quad f^{(r)} \mapsto f^{(r)} \otimes 1.$$

Finally, assuming that $s_{2,1} > 0$, let σ_- be the shift matrix obtained from σ by subtracting 1 from $s_{2,1}$. Then there is an algebra homomorphism

$$\Delta_- : Y_\sigma \to U(\mathfrak{gl}_1) \otimes Y_{\sigma_-}, \tag{2.16}$$

$$d_1^{(r)} \mapsto 1 \otimes d_1^{(r)}, \qquad d_2^{(r)} \mapsto 1 \otimes d_2^{(r)} + (-1)^{|2|} e_{1,1} \otimes d_2^{(r-1)},$$

$$f^{(r)} \mapsto 1 \otimes f^{(r)} + (-1)^{|2|} e_{1,1} \otimes f^{(r-1)}, \quad e^{(r)} \mapsto 1 \otimes e^{(r)}.$$

If $s_{1,2} > 0$, we denote $(\sigma^{\mathrm{up}})_+ = (\sigma_+)^{\mathrm{up}}$ by σ_+^{up}. If $s_{2,1} > 0$ we denote $(\sigma^{\mathrm{lo}})_- = (\sigma_-)^{\mathrm{lo}}$ by σ_-^{lo}. If both $s_{1,2} > 0$ and $s_{2,1} > 0$ we denote $(\sigma_+)_- = (\sigma_-)_+$ by $\sigma_\pm$.

Lemma 2.10. *Assuming that $s_{1,2} > 0$ in the first diagram, $s_{2,1} > 0$ in the second diagram, and both $s_{1,2} > 0$ and $s_{2,1} > 0$ in the final diagram, the following commute:*

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta_+"] \arrow[d, "\Delta"'] & Y_{\sigma_+} \otimes U(\mathfrak{gl}_1) \arrow[d, "\Delta\otimes \mathrm{id}"] \\ Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}} \arrow[r, "\mathrm{id}\otimes\Delta_+"] & Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma_+^{\mathrm{up}}} \otimes U(\mathfrak{gl}_1) \end{tikzcd} \tag{2.17}$$

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta"] \arrow[d, "\Delta_-"'] & Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}} \arrow[d, "\Delta_-\otimes \mathrm{id}"] \\ U(\mathfrak{gl}_1) \otimes Y_{\sigma_-} \arrow[r, "\mathrm{id}\otimes\Delta"] & U(\mathfrak{gl}_1) \otimes Y_{\sigma_-^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}} \end{tikzcd} \tag{2.18}$$

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta_+"] \arrow[d, "\Delta_-"'] & Y_{\sigma_+} \otimes U(\mathfrak{gl}_1) \arrow[d, "\Delta_-\otimes \mathrm{id}"] \\ U(\mathfrak{gl}_1) \otimes Y_{\sigma_-} \arrow[r, "\mathrm{id}\otimes\Delta_+"] & U(\mathfrak{gl}_1) \otimes Y_{\sigma_\pm} \otimes U(\mathfrak{gl}_1) \end{tikzcd} \tag{2.19}$$

Proof. Check on Drinfeld generators using (2.15)–(2.16) and Proposition 2.5. □

Remark 2.11. Writing $\varepsilon : U(\mathfrak{gl}_1) \to \mathbb{C}$ for the counit, the maps $(\mathrm{id}\,\bar{\otimes}\varepsilon) \circ \Delta_+$ and $(\varepsilon\bar{\otimes}\,\mathrm{id}) \circ \Delta_-$ are the natural inclusions $Y_\sigma \to Y_{\sigma_+}$ and $Y_\sigma \to Y_{\sigma_-}$, respectively. Hence the maps Δ_+ and Δ_- are injective.

3. Truncation

Let $\sigma = (s_{i,j})_{1\le i,j\le 2}$ be a shift matrix. Suppose also that we are given an integer $l \ge s_{2,1} + s_{1,2}$ and set

$$k := l - s_{2,1} - s_{1,2} \ge 0.$$

In view of Lemma 2.10, we can iterate Δ_+ a total of $s_{1,2}$ times, Δ_- a total of $s_{2,1}$ times, and Δ a total of $(k-1)$ times in any order that makes sense (when $k = 0$ this means we apply the counit ε once at the very end) to obtain a well-defined homomorphism

$$\Delta_\sigma^l : Y_\sigma \to U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes Y^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}}.$$

For example, if $\sigma = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ then

$$\begin{aligned} \Delta_\sigma^3 &= (\mathrm{id}\otimes\varepsilon\bar{\otimes}\,\mathrm{id}\otimes\mathrm{id}) \circ (\Delta_- \otimes \mathrm{id}\otimes\mathrm{id}) \circ (\Delta_+ \otimes \mathrm{id}) \circ \Delta_+, \\ \Delta_\sigma^4 &= (\mathrm{id}\otimes\Delta_+ \otimes \mathrm{id}) \circ (\Delta_- \otimes \mathrm{id}) \circ \Delta_+ = (\mathrm{id}\otimes\Delta_+ \otimes \mathrm{id}) \circ (\mathrm{id}\otimes\Delta_+) \circ \Delta_-, \\ \Delta_\sigma^5 &= (\Delta_- \otimes \mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}) \circ (\mathrm{id}\otimes\Delta_+ \otimes \mathrm{id}) \circ (\mathrm{id}\otimes\Delta_+) \circ \Delta \\ &= (\mathrm{id}\otimes\Delta \otimes \mathrm{id}\otimes\mathrm{id}) \circ (\Delta_- \otimes \mathrm{id}\otimes\mathrm{id}) \circ (\mathrm{id}\otimes\Delta_+) \circ \Delta_+. \end{aligned}$$

Let

$$U_\sigma^l := U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes U(\mathfrak{gl}_{1|1})^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}}, \tag{3.1}$$

viewed as a superalgebra using the usual sign convention. Recalling (2.5), we obtain a homomorphism

$$\mathrm{ev}^l_\sigma := (\mathrm{id}^{\otimes s_{2,1}} \otimes \mathrm{ev}^{\otimes k} \otimes \mathrm{id}^{\otimes s_{1,2}}) \circ \Delta^l_\sigma : Y_\sigma \to U^l_\sigma. \tag{3.2}$$

Let

$$Y^l_\sigma := \mathrm{ev}^l_\sigma(Y_\sigma) \subseteq U^l_\sigma. \tag{3.3}$$

This is the *shifted Yangian of level* l.

In the special case that $\sigma = 0$ we denote ev^l_σ, Y^l_σ and U^l_σ simply by ev^l, Y^l and U^l, respectively, so that $Y^l = \mathrm{ev}^l(Y) \subseteq U^l$. We call Y^l the *Yangian of level* l. Writing $\bar{e}^{[c]}_{i,j} := (-1)^{|i|} 1^{\otimes(c-1)} \otimes e_{i,j} \otimes 1^{\otimes(l-c)}$, we have simply that

$$\mathrm{ev}^l(t^{(r)}_{i,j}) = \sum_{1 < c_1 < \cdots < c_r \leq l} \sum_{1 \leq h_1, \dots, h_{r-1} \leq 2} \bar{e}^{[c_1]}_{i,h_1} \bar{e}^{[c_2]}_{h_1,h_2} \cdots \bar{e}^{[c_r]}_{h_{r-1},j} \tag{3.4}$$

for any $1 \leq i,j \leq 2$ and $r \geq 0$. In particular, $\mathrm{ev}^l(t^{(r)}_{i,j}) = 0$ for $r > l$. In the proof of [G, Theorem 1], Gow shows that the kernel of $\mathrm{ev}^l : Y \twoheadrightarrow Y^l$ is generated by $\{t^{(r)}_{i,j} \mid 1 \leq i,j \leq 2, r > l\}$, and moreover the images of the ordered supermonomials in the remaining elements $\{t^{(r)}_{i,j} \mid 1 \leq i,j \leq 2, 0 < r \leq l\}$ give a basis for Y^l. (Actually she proves this for all $Y(\mathfrak{gl}_{m|n})$ not just $Y(\mathfrak{gl}_{1|1})$.) The goal in this section is to prove analogues of these statements for Y_σ with $\sigma \neq 0$.

Let I^l_σ be the two-sided ideal of Y_σ generated by the elements $d^{(r)}_1$ for $r > k$.

Lemma 3.1. $I^l_\sigma \subseteq \ker \mathrm{ev}^l_\sigma$.

Proof. We need to show that $\mathrm{ev}^l_\sigma(d^{(r)}_1) = 0$ for all $r > k$. We calculate this by first applying all the maps Δ_+ and Δ_- to deduce that

$$\mathrm{ev}^l_\sigma(d^{(r)}_1) = 1^{\otimes s_{2,1}} \otimes \mathrm{ev}^k(d^{(r)}_1) \otimes 1^{\otimes s_{1,2}}.$$

Since $d^{(r)}_1 = t^{(r)}_{1,1}$, it is then clear from (3.4) that $\mathrm{ev}^k(d^{(r)}_1) = 0$ for $r > k$. $\square$

Proposition 3.2. *The ideal* I^l_σ *contains all of the following elements:*

$$\sum_{s_{1,2} < a \leq r} d^{(r-a)}_1 e^{(a)} \text{ for } r > s_{1,2} + k; \tag{3.5}$$

$$\sum_{s_{2,1} < b \leq r} f^{(b)} d^{(r-b)}_1 \text{ for } r > s_{2,1} + k; \tag{3.6}$$

$$d^{(r)}_2 + \sum_{\substack{s_{1,2} < a \\ s_{2,1} < b \\ a+b \leq r}} f^{(b)} d^{(r-a-b)}_1 e^{(a)} \text{ for } r > l. \tag{3.7}$$

Proof. Consider the algebra $Y_\sigma[\![u^{-1}]\!][u]$ of formal Laurent series in the variable u^{-1} with coefficients in Y_σ. For any such formal Laurent series $p = \sum_{r \leq N} p_r u^r$ we write $[p]_{\geq 0}$ for its polynomial part $\sum_{r=0}^N p_r u^r$. Also write $\equiv$ for congruence modulo $Y_\sigma[u] + u^{-1} I^l_\sigma [\![u^{-1}]\!]$, so $p \equiv 0$ means that the u^r-coefficients of p lie in I^l_σ

for all $r < 0$. Note that if $p \equiv 0, q \in Y_\sigma[u]$, then $pq \equiv 0$. In this notation, we have by definition of I_σ^l that $u^k d_1(u) \equiv 0$. Introduce the power series

$$e_\sigma(u) := \sum_{r>s_{1,2}} e^{(r)}u^{-r}, \qquad f_\sigma(u) := \sum_{r>s_{2,1}} f^{(r)}u^{-r}.$$

The proposition is equivalent to the following assertions:

$$u^{s_{1,2}+k} d_1(u) e_\sigma(u) \equiv 0, \tag{3.8}$$

$$u^{s_{2,1}+k} f_\sigma(u) d_1(u) \equiv 0, \tag{3.9}$$

$$u^l \left(d_2(u) + f_\sigma(u) d_1(u) e_\sigma(u)\right) \equiv 0. \tag{3.10}$$

For the first two, we use the identities

$$(-1)^{|1|}[d_1(u), e^{(s_{1,2}+1)}] = u^{s_{1,2}} d_1(u) e_\sigma(u), \tag{3.11}$$

$$(-1)^{|1|}[f^{(s_{2,1}+1)}, d_1(u)] = u^{s_{2,1}} f_\sigma(u) d_1(u). \tag{3.12}$$

These are easily checked by considering the u^{-r}-coefficients on each side and using the relations in Theorem 2.2. Assertions (3.8)–(3.9) follow from (3.11)–(3.12) on multiplying by u^k as $u^k d_1(u) \equiv 0$. For the final assertion (3.10), recall the elements $c^{(r)}$ from (2.10). Let $c_\sigma(u) := \sum_{r>s_{2,1}+s_{1,2}} c^{(r)}u^{-r}$. Another routine check using the relations shows that

$$(-1)^{|1|}[f^{(s_{2,1}+1)}, e_\sigma(u)] = u^{s_{2,1}} c_\sigma(u). \tag{3.13}$$

Using (3.8) and (3.12)–(3.13) we deduce that

$$\begin{aligned} 0 &\equiv (-1)^{|1|} u^{s_{1,2}+k}[f^{(s_{2,1}+1)}, d_1(u) e_\sigma(u)] \\ &= u^{s_{1,2}+k} d_1(u)(-1)^{|1|}[f^{(s_{2,1}+1)}, e_\sigma(u)] + u^{s_{1,2}+k}(-1)^{|1|}[f^{(s_{2,1}+1)}, d_1(u)] e_\sigma(u) \\ &= u^l d_1(u) c_\sigma(u) + u^l f_\sigma(u) d_1(u) e_\sigma(u). \end{aligned}$$

To complete the proof of (3.10), it remains to observe that

$$u^{s_{2,1}+s_{1,2}} c_\sigma(u) = u^{s_{2,1}+s_{1,2}} \tilde{d}_1(u) d_2(u) - [u^{s_{2,1}+s_{1,2}} \tilde{d}_1(u) d_2(u)]_{\geq 0}$$

hence $u^l d_1(u) c_\sigma(u) \equiv u^l d_2(u)$. □

For the rest of the section, we fix some total ordering on the set

$$\begin{aligned} \Omega := \{d_1^{(r)} \mid 0 < r \leq k\} \cup \{d_2^{(r)} \mid 0 < r \leq l\} \\ \cup \{e^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}. \end{aligned} \tag{3.14}$$

Lemma 3.3. *The quotient algebra Y_σ/I_σ^l is spanned by the images of the ordered supermonomials in the elements of Ω.*

Proof. The Kazhdan filtration on Y_σ induces a filtration on Y_σ/I_σ^l with respect to which $\mathrm{gr}(Y_\sigma/I_\sigma^l)$ is a graded quotient of $\mathrm{gr}\, Y_\sigma$. We know already that $\mathrm{gr}\, Y_\sigma$ is supercommutative, hence so too is $\mathrm{gr}(Y_\sigma/I_\sigma^l)$. Let $\underline{d}_i^{(r)} := \mathrm{gr}_r(d_i^{(r)} + I_\sigma^l), \underline{e}^{(r)} := \mathrm{gr}_r(e^{(r)} + I_\sigma^l)$ and $\underline{f}^{(r)} := \mathrm{gr}_r(f^{(r)} + I_\sigma^l)$.

To prove the lemma it is enough to show that $\mathrm{gr}(Y_\sigma/I_\sigma^l)$ is generated by $\{\underline{d}_1^{(r)} \mid 0 < r \leq k\} \cup \{\underline{d}_2^{(r)} \mid 0 < r \leq l\} \cup \{\underline{e}^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{\underline{f}^{(r)} \mid s_{2,1} <$

$r \le s_{2,1}+k\}$. This follows because $\underline{d}_1^{(r)} = 0$ for $r > k$, and each of the elements $\underline{d}_2^{(r)}$ $(r > l)$, $\underline{e}^{(r)}$ $(r > s_{1,2}+k)$ and $\underline{f}^{(r)}$ $(r > s_{2,1}+k)$ can be expressed as polynomials in generators of strictly smaller degrees by Proposition 3.2. □

Lemma 3.4. *The image under* ev_σ^l *of the ordered supermonomials in the elements of* Ω *are linearly independent in* Y_σ^l*.*

Proof. Consider the standard filtration on U_σ^l generated by declaring that all the elements of the form $1\otimes\cdots\otimes 1\otimes x\otimes 1\otimes\cdots\otimes 1$ for $x \in \mathfrak{gl}_1$ or $\mathfrak{gl}_{1|1}$ are in degree 1. It induces a filtration on Y_σ^l so that $\operatorname{gr} Y_\sigma^l$ is a graded subalgebra of $\operatorname{gr} U_\sigma^l$. Note that $\operatorname{gr} U_\sigma^l$ is supercommutative, hence so is the subalgebra $\operatorname{gr} Y_\sigma^l$. Each of the elements $\mathrm{ev}_\sigma^l(d_i^{(r)})$, $\mathrm{ev}_\sigma^l(e^{(r)})$ and $\mathrm{ev}_\sigma^l(f^{(r)})$ are in filtered degree r by the definition of ev_σ^l. Let $\underline{d}_i^{(r)} := \operatorname{gr}_r(\mathrm{ev}_\sigma^l(d_i^{(r)}))$, $\underline{e}^{(r)} := \operatorname{gr}_r(\mathrm{ev}_\sigma^l(e^{(r)}))$ and $\underline{f}^{(r)} := \operatorname{gr}_r(\mathrm{ev}_\sigma^l(f^{(r)}))$.

Let M be the set of ordered supermonomials in $\{\underline{d}_1^{(r)} \mid 0 < r \le k\} \cup \{\underline{d}_2^{(r)} \mid 0 < r \le l\} \cup \{\underline{e}^{(r)} \mid s_{1,2} < r \le s_{1,2}+k\} \cup \{\underline{f}^{(r)} \mid s_{2,1} < r \le s_{2,1}+k\}$. To prove the lemma, it suffices to show that M is linearly independent in $\operatorname{gr} Y_\sigma^l$. For this, we proceed by induction on $s_{2,1}+s_{1,2}$.

To establish the base case $s_{2,1}+s_{1,2} = 0$, i.e. $\sigma = 0$, $Y_\sigma = Y$ and $Y_\sigma^l = Y^l$. Let $\underline{t}_{i,j}^{(r)}$ denote $\operatorname{gr}_r(\mathrm{ev}_\sigma^l(t_{i,j}^{(r)}))$. Fix a total order on $\{\underline{t}_{i,j}^{(r)} \mid 1 \le i,j \le 2, 0 < r \le l\}$ and let M' be the resulting set of ordered supermonomials. Exploiting the explicit formula (3.4), Gow shows in the proof of [G, Theorem 1] that M' is linearly independent. By (2.6)–(2.9), any element of M is a linear combination of elements of M' of the same degree, and vice versa. So we deduce that M is linearly independent too.

For the induction step, suppose that $s_{2,1}+s_{1,2} > 0$. Then we either have $s_{2,1} > 0$ or $s_{1,2} > 0$. We just explain the argument for the latter case; the proof in the former case is entirely similar replacing Δ_+ with Δ_-. Recall that σ_+ denotes the shift matrix obtained from σ by subtracting 1 from $s_{1,2}$. So $U_\sigma^l = U_{\sigma_+}^{l-1} \otimes U(\mathfrak{gl}_1)$. By its definition, we have that $\mathrm{ev}_\sigma^l = (\mathrm{ev}_{\sigma_+}^{l-1} \otimes \mathrm{id}) \circ \Delta_+$, hence $Y_\sigma^l \subseteq Y_{\sigma_+}^{l-1} \otimes U(\mathfrak{gl}_1)$. Let $x := \operatorname{gr}_1 e_{1,1} \in \operatorname{gr} U(\mathfrak{gl}_1)$. Then

$$\underline{d}_1^{(r)} = \dot{\underline{d}}_1^{(r)} \otimes 1, \qquad \underline{d}_2^{(r)} = \dot{\underline{d}}_2^{(r)} \otimes 1 + (-1)^{|2|} \dot{\underline{d}}_2^{(r-1)} \otimes x,$$
$$\underline{f}^{(r)} = \dot{\underline{f}}^{(r)} \otimes 1, \qquad \underline{e}^{(r)} = \dot{\underline{e}}^{(r)} \otimes 1 + (-1)^{|2|} \dot{\underline{e}}^{(r-1)} \otimes x.$$

The notation is potentially confusing here so we have decorated elements of $\operatorname{gr} Y_{\sigma_+}^{l-1} \subseteq \operatorname{gr} U_{\sigma_+}^{l-1}$ with a dot. It remains to observe from the induction hypothesis applied to $\operatorname{gr} Y_{\sigma_+}^{l-1}$ that ordered supermonomials in $\{\dot{\underline{d}}_1^{(r)} \otimes 1 \mid 0 < r \le k\} \cup \{\dot{\underline{d}}_2^{(r-1)} \otimes x \mid 0 < r \le l\} \cup \{\dot{\underline{e}}^{(r-1)} \otimes x \mid s_{1,2} < r \le s_{1,2}+k\} \cup \{\dot{\underline{f}}^{(r)} \otimes 1 \mid 0 < r < s_{1,2}+k\}$ are linearly independent. □

Theorem 3.5. *The kernel of* $\mathrm{ev}_\sigma^l : Y_\sigma \to Y_\sigma^l$ *is equal to the two-sided ideal* I_σ^l *generated by the elements* $\{d_1^{(r)} \mid r > k\}$*. Hence* ev_σ^l *induces an algebra isomorphism between* Y_σ/I_σ^l *and* Y_σ^l*.*

Proof. By Lemma 3.1, ev_σ^l induces a surjection $Y_\sigma/I_\sigma^l \twoheadrightarrow Y_\sigma^l$. It maps the spanning set from Lemma 3.3 onto the linearly independent set from Lemma 3.4. Hence it is an isomorphism and both sets are actually bases. $\square$

Henceforth we will *identify* Y_σ^l with the quotient Y_σ/I_σ^l, and we will abuse notation by denoting the canonical images in Y_σ^l of the elements $d_i^{(r)}, e^{(r)} \dots$ of Y_σ by the same symbols $d_i^{(r)}, e^{(r)}, \dots$. This will not cause any confusion as we will not work with Y_σ again.

Here is the PBW theorem for Y_σ^l, which was noted already in the proof of Theorem 3.5.

Corollary 3.6. *Order the set* $\{d_1^{(r)} \mid 0 < r \leq k\} \cup \{d_2^{(r)} \mid 0 < r \leq l\} \cup \{e^{(r)} \mid s_{1,2} < r \leq s_{1,2}+k\} \cup \{f^{(r)} \mid s_{2,1} < r \leq s_{2,1}+k\}$ *in some way. The ordered supermonomials in these elements give a basis for* Y_σ^l.

Remark 3.7. In the arguments in this section, we have defined *two* filtrations on Y_σ^l, one in the proof of Lemma 3.3 induced by the Kazhdan filtration on Y_σ, the other in the proof of Lemma 3.4 induced by the standard filtration on U_σ^l. Using Corollary 3.6, one can check that these two filtrations coincide.

Remark 3.8. Theorem 3.5 shows that Y_σ^l has generators

$$\{d_i^{(r)} \mid i = 1, 2, r > 0\} \cup \{e^{(r)} \mid r > s_{1,2}\} \cup \{f^{(r)} \mid r > s_{2,1}\}$$

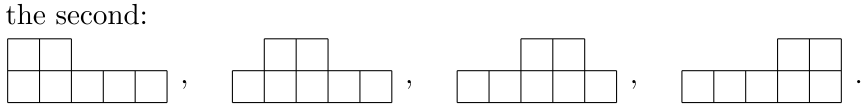
subject only to the relations from Theorem 2.2 and the additional truncation relations $d_1^{(r)} = 0$ for $r > k$. Corollary 3.6 shows that all but finitely many of the generators are redundant. In special cases it is possible to optimize the relations too. For example, if $l = s_{2,1} + s_{1,2} + 1$ and we set $d := d_1^{(1)}, e := e^{(s_{1,2}+1)}, f := f^{(s_{2,1}+1)}$, then Y_σ^l is generated by its even central elements $c^{(1)}, \dots, c^{(l)}$ from (2.10), the even element d, and the odd elements e, f, subject only to the relations

$$[d,e] = (-1)^{|1|}e, \qquad [d,f] = -(-1)^{|1|}f, \qquad [e,f] = (-1)^{|1|}c^{(l)},$$
$$[c^{(r)}, c^{(s)}] = [c^{(r)}, d] = [c^{(r)}, e] = [c^{(r)}, f] = [e,e] = [f,f] = 0,$$

for $r, s = 1, \dots, l$. To see this, observe that these elements generate Y_σ^l and they satisfy the given relations, then apply Corollary 3.6.

4. Principal W-algebras

We turn to the W-algebra side of the story. Let π be a (two-rowed) *pyramid*, that is, a collection of boxes in the plane arranged in two connected rows such that each box in the first (top) row lies directly above a box in the second (bottom) row. For example, here are all the pyramids with two boxes in the first row and five in the second:



Let k (resp. l) denote the number of boxes in the first (resp. second) row of π, so that $k \leq l$. The parity sequence fixed in (2.1) allows us to talk about the parities of the rows of π: the ith row is of parity $|i|$. Let m be the number of boxes in the even row, i.e. the row with parity $\bar{0}$, and n be the number of boxes in the odd row, i.e. the row with parity $\bar{1}$. Then label the boxes in the even (resp. odd) row from left to right by the numbers $1, \dots, m$ (resp. $m+1, \dots, m+n$). For example here is one of the above pyramids with boxes labelled in this way assuming that $(|1|, |2|) = (\bar{1}, \bar{0})$, i.e. the bottom row is even and the top row is odd:

$$\begin{array}{ccccc} & \boxed{6} & \boxed{7} & & \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \end{array}\ . \tag{4.1}$$

Numbering the columns of π by $1, \dots, l$ in order from left to right, we write $\mathrm{row}(i)$ and $\mathrm{col}(i)$ for the row and column numbers of the ith box in this labelling.

Now let $\mathfrak{g} := \mathfrak{gl}_{m|n}(\mathbb{C})$ for m and n coming from the pyramid π and the fixed parity sequence as in the previous paragraph. Let $\mathfrak{t}$ be the Cartan subalgebra consisting of all diagonal matrices and $\varepsilon_1, \dots, \varepsilon_{m+n} \in \mathfrak{t}^*$ be the basis such that $\varepsilon_i(e_{j,j}) = \delta_{i,j}$ for each $j = 1, \dots, m+n$. The *supertrace form* $(.|.)$ on $\mathfrak{g}$ is the non-degenerate invariant supersymmetric bilinear form defined from $(x|y) = \mathrm{str}(xy)$, where the supertrace $\mathrm{str}\, A$ of matrix $A = (a_{i,j})_{1 \le i,j \le m+n}$ means $a_{1,1} + \cdots + a_{m,m} - a_{m+1,m+1} - \cdots - a_{m+n,m+n}$. It induces a bilinear form $(.|.)$ on $\mathfrak{t}^*$ such that $(\varepsilon_i|\varepsilon_j) = (-1)^{|\,\mathrm{row}(i)|}\delta_{i,j}$.

We have the explicit principal nilpotent element

$$e := \sum_{i,j} e_{i,j} \in \mathfrak{g}_{\bar{0}} \tag{4.2}$$

summing over all adjacent pairs $\boxed{i}\boxed{j}$ of boxes in the pyramid π. In the example above, we have that $e = e_{1,2} + e_{2,3} + e_{3,4} + e_{4,5} + e_{6,7}$. Let $\chi \in \mathfrak{g}^*$ be defined by $\chi(x) := (x|e)$. If we set

$$\bar{e}_{i,j} := (-1)^{|\,\mathrm{row}(i)|} e_{i,j}, \tag{4.3}$$

then we have that

$$\chi(\bar{e}_{i,j}) = \begin{cases} 1 & \text{if } \boxed{j}\boxed{i} \text{ is an adjacent pair of boxes in } \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Introduce a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}(r)$ by declaring that $e_{i,j}$ is of degree

$$\deg(e_{i,j}) := \mathrm{col}(j) - \mathrm{col}(i). \tag{4.5}$$

This is a *good grading* for e, which means that $e \in \mathfrak{g}(1)$ and the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ is contained in $\bigoplus_{r \ge 0} \mathfrak{g}(r)$; see [H] for more about good gradings on Lie superalgebras (one should double the degrees of our grading to agree with the terminology there). Set

$$\mathfrak{p} := \bigoplus_{r \ge 0} \mathfrak{g}(r) \qquad \mathfrak{h} := \mathfrak{g}(0), \qquad \mathfrak{m} := \bigoplus_{r < 0} \mathfrak{g}(r).$$

Note that χ restricts to a character of $\mathfrak{m}$. Let $\mathfrak{m}_\chi := \{x - \chi(x) \mid x \in \mathfrak{m}\}$, which is a shifted copy of $\mathfrak{m}$ inside $U(\mathfrak{m})$. Then the *principal W-algebra* associated to the pyramid π is

$$W_\pi := \{u \in U(\mathfrak{p}) \mid u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})\}. \tag{4.6}$$

It is straightforward to check that W_π is a subalgebra of $U(\mathfrak{p})$.

The first important result about W_π is its *PBW theorem*. This is noted already in [Z, Remark 3.10], where it is described for arbitrary basic classical Lie superalgebras modulo a mild assumption on e (which is trivially satisfied here).

To formulate the result precisely, embed e into an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}_{\bar 0}$, such that $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-1)$. It follows from $\mathfrak{sl}_2$ representation theory that

$$\mathfrak{p} = \mathfrak{g}^e \oplus [\mathfrak{p}^\perp, f], \tag{4.7}$$

where $\mathfrak{p}^\perp = \bigoplus_{r>0} \mathfrak{g}(r)$ denotes the nilradical of $\mathfrak{p}$. Also introduce the *Kazhdan filtration* on $U(\mathfrak{p})$, which is generated by declaring for each $r \geq 0$ that $x \in \mathfrak{g}(r)$ is of Kazhdan degree $r + 1$. The associated graded superalgebra $\operatorname{gr} U(\mathfrak{p})$ is supercommutative and is naturally identified with the symmetric superalgebra $S(\mathfrak{p})$, viewed as a positively graded algebra via the analogously defined *Kazhdan grading*. The Kazhdan filtration on $U(\mathfrak{p})$ induces a Kazhdan filtration on $W_\pi \subseteq U(\mathfrak{p})$ so that $\operatorname{gr} W_\pi \subseteq \operatorname{gr} U(\mathfrak{p}) = S(\mathfrak{p})$.

Theorem 4.1. *Let $p : S(\mathfrak{p}) \to S(\mathfrak{g}^e)$ be the homomorphism induced by the projection of $\mathfrak{p}$ onto $\mathfrak{g}^e$ along (4.7). The restriction of p defines an isomorphism of Kazhdan-graded superalgebras $\operatorname{gr} W_\pi \xrightarrow{\sim} S(\mathfrak{g}^e)$.*

Proof. Superize the arguments in [GG] as suggested in [Z, Remark 3.10]. □

In order to apply Theorem 4.1, it is helpful to have available an explicit basis for the centralizer $\mathfrak{g}^e$. We say that a shift matrix $\sigma = (s_{i,j})_{1 \leq i,j \leq 2}$ is *compatible with* π if either $k > 0$ and π has $s_{2,1}$ columns of height one on its left side and $s_{1,2}$ columns of height one on its right side, or if $k = 0$ and $l = s_{2,1} + s_{1,2}$. These conditions determine a unique shift matrix σ when $k > 0$, but there is some minor ambiguity if $k = 0$ (which should never cause any concern). For example if π is as in (4.1) then $\sigma = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ is the only compatible shift matrix.

Lemma 4.2. *Let $\sigma = (s_{i,j})_{1 \leq i,j \leq 2}$ be a shift matrix compatible with π. For $r \geq 0$, let*

$$x_{i,j}^{(r)} := \sum_{\substack{1 \leq p,q \leq m+n \\ \mathrm{row}(p)=i, \mathrm{row}(q)=j \\ \deg(e_{p,q})=r-1}} \bar{e}_{p,q} \in \mathfrak{g}(r-1).$$

Then the elements $\{x_{1,1}^{(r)} \mid 0 < r \leq k\} \cup \{x_{2,2}^{(r)} \mid 0 < r \leq l\} \cup \{x_{1,2}^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{x_{2,1}^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}$ give a homogeneous basis for $\mathfrak{g}^e$.

Proof. As e is even, the centralizer of e in $\mathfrak{g}$ is just the same as a vector space as the centralizer of e viewed as an element of $\mathfrak{gl}_{m+n}(\mathbb{C})$, so this follows as a special case of [BK1, Lemma 7.3] (which is [SS, IV.1.6]). □

We come to the key ingredient in our approach: the explicit definition of special elements of $U(\mathfrak{p})$ some of which turn out to generate W_π. Define another ordering $\prec$ on the set $\{1, \dots, m+n\}$ by declaring that $i \prec j$ if $\mathrm{col}(i) < \mathrm{col}(j)$, or if $\mathrm{col}(i) = \mathrm{col}(j)$ and $\mathrm{row}(i) < \mathrm{row}(j)$. Let $\tilde\rho \in \mathfrak{t}^*$ be the weight with

$$(\tilde\rho|\varepsilon_j) = \#\left\{ i \,\middle|\, i \preceq j \text{ and } |\mathrm{row}(i)| = \bar 1 \right\} - \#\left\{ i \,\middle|\, i \prec j \text{ and } |\mathrm{row}(i)| = \bar 0 \right\}. \tag{4.8}$$

For example, if π is as in (4.1), then $\tilde\rho = -\varepsilon_4 - 2\varepsilon_5$. The weight $\tilde\rho$ extends to a character of $\mathfrak{p}$, so there are automorphisms

$$S_{\pm\tilde\rho} : U(\mathfrak{p}) \to U(\mathfrak{p}), \qquad e_{i,j} \mapsto e_{i,j} \pm \delta_{i,j} \tilde\rho(e_{i,i}). \tag{4.9}$$

Finally, given $1 \leq i, j \leq 2, 0 \leq \varsigma \leq 2$ and $r \geq 1$, we define

$$t_{i,j;\varsigma}^{(r)} := S_{\tilde{\rho}} \left(\sum_{s=1}^{r} (-1)^{r-s} \sum_{\substack{i_1,\dots,i_s \\ j_1,\dots,j_s}} (-1)^{\#\{a=1,\dots,s-1 \mid \mathrm{row}(j_a) \leq \varsigma\}} \bar{e}_{i_1,j_1} \cdots \bar{e}_{i_s,j_s} \right), \quad (4.10)$$

where the sum is over all $1 \leq i_1, \dots, i_s, j_1, \dots, j_s \leq m+n$ such that

- $\mathrm{row}(i_1) = i$ and $\mathrm{row}(j_s) = j$;
- $\mathrm{col}(i_a) \leq \mathrm{col}(j_a)$ $(a = 1, \dots, s)$;
- $\mathrm{row}(i_{a+1}) = \mathrm{row}(j_a)$ $(a = 1, \dots, s-1)$;
- if $\mathrm{row}(j_a) > \varsigma$ then $\mathrm{col}(i_{a+1}) > \mathrm{col}(j_a)$ $(a = 1, \dots, s-1)$;
- if $\mathrm{row}(j_a) \leq \varsigma$ then $\mathrm{col}(i_{a+1}) \leq \mathrm{col}(j_a)$ $(a = 1, \dots, s-1)$;
- $\deg(e_{i_1,j_1}) + \cdots + \deg(e_{i_s,j_s}) = r - s$.

It is convenient to collect these elements together into the generating function

$$t_{i,j;\varsigma}(u) := \sum_{r \geq 0} t_{i,j;\varsigma}^{(r)} u^{-r} \in U(\mathfrak{p})[\![u^{-1}]\!], \quad (4.11)$$

setting $t_{i,j;\varsigma}^{(0)} := \delta_{i,j}$. The following two propositions should already convince the reader of the remarkable nature of these elements.

Proposition 4.3. *The following identities hold in* $U(\mathfrak{p})[\![u^{-1}]\!]$*:*

$$t_{1,1;1}(u) = t_{1,1;0}(u)^{-1}, \quad (4.12)$$
$$t_{2,2;2}(u) = t_{2,2;1}(u)^{-1}, \quad (4.13)$$
$$t_{1,2;0}(u) = t_{1,1;0}(u) t_{1,2;1}(u), \quad (4.14)$$
$$t_{2,1;0}(u) = t_{2,1;1}(u) t_{1,1;0}(u), \quad (4.15)$$
$$t_{2,2;0}(u) = t_{2,2;1}(u) + t_{2,1;1}(u) t_{1,1;0}(u) t_{1,2;1}(u). \quad (4.16)$$

Proof. This is proved in [BK1, Lemma 9.2]; the argument there is entirely formal and does not depend on the underlying associative algebra in which the calculations are performed. □

Proposition 4.4. *Let σ be a shift matrix compatible with π. The following elements of $U(\mathfrak{p})$ belong to W_π: all $t_{1,1;0}^{(r)}, t_{1,1;1}^{(r)}, t_{2,2;1}^{(r)}$ and $t_{2,2;2}^{(r)}$ for $r > 0$; all $t_{1,2;1}^{(r)}$ for $r > s_{1,2}$; all $t_{2,1;1}^{(r)}$ for $r > s_{2,1}$.*

Proof. Postponed to the next section. □

Now we can deduce our main result. For any shift matrix σ compatible with π, we identify $U(\mathfrak{h})$ with the algebra U_σ^l from (3.1) so that

$$e_{i,j} \equiv \begin{cases} 1^{\otimes(c-1)} \otimes e_{\mathrm{row}(i),\mathrm{row}(j)} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 2, \\ 1^{\otimes(c-1)} \otimes e_{1,1} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 1, \end{cases}$$

for any $1 \leq i, j \leq m+n$ with $c := \mathrm{col}(i) = \mathrm{col}(j)$, where q_c denotes the number of boxes in this column of π. Define the *Miura transform*

$$\mu : W_\pi \to U(\mathfrak{h}) = U_\sigma^l \quad (4.17)$$

to be the restriction to W_π of the shift automorphism $S_{-\tilde{\rho}}$ composed with the natural homomorphism $\mathrm{pr} : U(\mathfrak{p}) \to U(\mathfrak{h})$ induced by the projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$.

Theorem 4.5. *Let σ be a shift matrix compatible with π. The Miura transform is injective and its image is the algebra $Y_\sigma^l \subseteq U_\sigma^l$ from (3.3). Hence it defines a superalgebra isomorphism*

$$\mu : W_\pi \xrightarrow{\sim} Y_\sigma^l \tag{4.18}$$

between W_π and the shifted Yangian of level l. Moreover μ maps the invariants from Proposition 4.4 to the Drinfeld generators of Y_σ^l as follows:

$$\mu(t_{1,1;0}^{(r)}) = d_1^{(r)} \qquad (r > 0), \qquad \mu(t_{1,1;1}^{(r)}) = \tilde{d}_1^{(r)} \qquad (r > 0), \tag{4.19}$$

$$\mu(t_{2,2;1}^{(r)}) = d_2^{(r)} \qquad (r > 0), \qquad \mu(t_{2,2;2}^{(r)}) = \tilde{d}_2^{(r)} \qquad (r > 0), \tag{4.20}$$

$$\mu(t_{1,2;1}^{(r)}) = e^{(r)} \qquad (r > s_{1,2}), \qquad \mu(t_{2,1;1}^{(r)}) = f^{(r)} \qquad (r > s_{2,1}). \tag{4.21}$$

Proof. We first establish the identities (4.19)–(4.21). Note that the identities involving $\tilde{d}_i^{(r)}$ are consequences of the ones involving $d_i^{(r)}$ thanks to (4.12)–(4.13), recalling also that $\tilde{d}_i(u) = d_i(u)^{-1}$. To prove all the other identities, we proceed by induction on $s_{2,1} + s_{1,2} = l - k$.

First consider the base case $l = k$. For $1 \leq i, j \leq 2$ and $r > 0$ we know in this situation that $t_{i,j;0}^{(r)} \in W_\pi$ since, using (4.14)–(4.16), it can be expanded in terms of elements all of which are known to lie in W_π by Proposition 4.4; see also Lemma 5.1 below. Moreover, we have directly from (4.10) and (3.4) that $\mu(t_{i,j;0}^{(r)}) = t_{i,j}^{(r)} \in Y_\sigma^l$. Hence $\mu(t_{i,j;0}(u)) = t_{i,j}(u)$. The result follows from this, (2.6)–(2.7), and the analogous expressions for $t_{1,1;0}(u), t_{2,2;1}(u), t_{1,2;1}(u)$ and $t_{2,1;1}(u)$ derived from (4.14)–(4.16).

Now consider the induction step, so $s_{2,1} + s_{1,2} > 0$. There are two cases according to whether $s_{2,1} > 0$ or $s_{1,2} > 0$. We just explain the argument for the latter situation, since the former is entirely similar. Let $\dot{\pi}$ be the pyramid obtained from π by removing the rightmost column and let $W_{\dot{\pi}}$ be the corresponding finite W-algebra. We denote its Miura transform by $\dot{\mu} : W_{\dot{\pi}} \to U_{\sigma_+}^{l-1}$, and similarly decorate all other notation related to $\dot{\pi}$ with a dot to avoid confusion. Now we proceed to show that $\mu(t_{1,2;1}^{(r)}) = e^{(r)}$ for each $r > s_{1,2}$. By induction, we know that $\dot{\mu}(\dot{t}_{1,2;1}^{(r)}) = \dot{e}^{(r)}$ for each $r \geq s_{1,2}$. But then it follows from the explicit form of (4.10), together with (2.15) and the definition of the evaluation homomorphism (3.2), that

$$\begin{aligned}
\mu(t_{1,2;1}^{(r)}) &= \dot{\mu}\big(\dot{t}_{1,2;1}^{(r)}\big) \otimes 1 + (-1)^{|2|} \dot{\mu}\big(\dot{t}_{1,2;1}^{(r-1)}\big) \otimes e_{1,1} \\
&= \dot{e}^{(r)} \otimes 1 + (-1)^{|2|} \dot{e}^{(r-1)} \otimes e_{1,1} = e^{(r)},
\end{aligned}$$

providing $r > s_{1,2}$. The other cases are similar.

Now we deduce the rest of the theorem from (4.19)–(4.21). Order the elements of the set

$$\begin{aligned}
\Omega := \{t_{1,1;0}^{(r)} \mid 0 < r \leq k\} \cup \{t_{2,2;1}^{(r)} \mid 0 < r \leq l\} \\
\cup \{t_{1,2;1}^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{t_{2,1;1}^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}
\end{aligned}$$

in some way. By Proposition 4.4, each $t_{i,j;\varsigma}^{(r)} \in \Omega$ belongs to W_π. Moreover, from the definition (4.10), it is in filtered degree r and $\mathrm{gr}_r\, t_{i,j;\varsigma}^{(r)}$ is equal up to a sign to the element $x_{i,j}^{(r)}$ from Lemma 4.2 plus a linear combination of monomials in elements of strictly smaller Kazhdan degree. Using Theorem 4.1, we deduce that the set of all ordered supermonomials in the set Ω gives a linear basis for W_π. By (4.19)–(4.21) and Corollary 3.6, μ maps this basis onto a basis for $Y_\sigma^l \subseteq U_\sigma^l$. Hence μ is an isomorphism. $\square$

Remark 4.6. The grading $\mathfrak{p} = \bigoplus_{r \geq 0} \mathfrak{g}(r)$ induces a grading on the superalgebra $U(\mathfrak{p})$. However W_π is not a graded subalgebra. Instead, we get induced another filtration on W_π, with respect to which the associated graded superalgebra $\mathrm{gr}' W_\pi$ is identified with a graded subalgebra of $U(\mathfrak{p})$. Each of the invariants $t_{i,j;\varsigma}^{(r)}$ from Proposition 4.4 belongs to filtered degree $(r-1)$ and has image $(-1)^{r-1} x_{i,j}^{(r)}$ in the associated graded algebra. Combined with Lemma 4.2 and the usual PBW theorem for $\mathfrak{g}^e$, it follows that $\mathrm{gr}' W_\pi = U(\mathfrak{g}^e)$. Moreover this filtration on W_π corresponds under the isomorphism μ to the filtration on Y_σ^l induced by the Lie filtration on Y_σ.

Remark 4.7. In this section, we have worked with the "right-handed" definition (4.6) of the finite W-algebra. One can also consider the "left-handed" version

$$W_\pi^\dagger := \{u \in U(\mathfrak{p}) \mid \mathfrak{m}_\chi u \subseteq U(\mathfrak{g})\mathfrak{m}_\chi\}.$$

There is an analogue of Theorem 4.5 for $W_\pi^\dagger$, via which one sees that $W_\pi \cong W_\pi^\dagger$. More precisely, we define the "left-handed" Miura transform $\mu^\dagger : W_\pi^\dagger \to U(\mathfrak{h})$ as above but twisting with the shift automorphism $S_{-\tilde{\rho}^\dagger}$ rather than $S_{-\tilde{\rho}}$, where

$$(\tilde{\rho}^\dagger|\varepsilon_j) = \#\left\{ i \,\middle|\, i \preceq^\dagger j \text{ and } |\operatorname{row}(i)| = \bar{1} \right\} - \#\left\{ i \,\middle|\, i \prec^\dagger j \text{ and } |\operatorname{row}(i)| = \bar{0} \right\} \tag{4.22}$$

and $i \prec^\dagger j$ means either $\operatorname{col}(i) > \operatorname{col}(j)$, or $\operatorname{col}(i) = \operatorname{col}(j)$ and $\operatorname{row}(i) < \operatorname{row}(j)$. The analogue of Theorem 4.5 asserts that $\mu^\dagger$ is injective with the same image as μ. Hence $\mu^{-1} \circ \mu^\dagger$, i.e. the restriction of the shift $S_{\tilde{\rho}-\tilde{\rho}^\dagger} : U(\mathfrak{p}) \to U(\mathfrak{p})$, gives an isomorphism between $W_\pi^\dagger$ and W_π. Noting that

$$\tilde{\rho} - \tilde{\rho}^\dagger = \sum_{\substack{1 \leq i,j \leq m+n \\ \operatorname{col}(i) < \operatorname{col}(j)}} (-1)^{|\operatorname{row}(i)| + |\operatorname{row}(j)|} (\varepsilon_i - \varepsilon_j), \tag{4.23}$$

there is a more conceptual explanation for this isomorphism along the lines of the proof given in the non-super case in [BGK, Corollary 2.9].

Remark 4.8. Another consequence of Theorem 4.5 together with Remarks 2.9 and 2.1 is that up to isomorphism the algebra W_π depends only on the set $\{m, n\}$, i.e. on the isomorphism type of $\mathfrak{g}$, not on the particular choice of the pyramid π or the parity sequence. As observed in [Z, Remark 3.10], this can also be proved by mimicking [BG, Theorem 2].

5. Proof of invariance

In this section, we prove Proposition 4.4. We keep all notation as in the statement of the proposition. Showing that $u \in U(\mathfrak{p})$ lies in the algebra W_π is equivalent to showing that $[x,u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $x \in \mathfrak{m}$, or even just for all x in a set of generators for $\mathfrak{m}$. Let

$$\Omega := \{t_{1,1;0}^{(r)} \mid r > 0\} \cup \{t_{1,2;1}^{(r)} \mid r > s_{1,2}\} \cup \{t_{2,1;1}^{(r)} \mid r > s_{2,1}\} \cup \{t_{2,2;1}^{(r)} \mid r > 0\}. \tag{5.1}$$

Our goal is to show that $[x,u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for x running over a set of generators of $\mathfrak{m}$ and $u \in \Omega$. Proposition 4.4 follows from this since all the other elements listed in the statement of the proposition can be expressed in terms of elements of Ω thanks to Proposition 4.3. Also observe for the present purposes that there is some freedom in the choice of the weight $\tilde{\rho}$: it can be adjusted by adding on any multiple of "supertrace" $\varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$. This just twists the elements $t_{i,j;\varsigma}^{(r)}$ by an automorphism of $U(\mathfrak{g})$ so does not have any effect on whether they belong to W_π. So sometimes in this section we will allow ourselves to change the choice of $\tilde{\rho}$.

Lemma 5.1. *Assuming $k = l$, we have that $[x, t_{i,j;0}^{(r)}] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $x \in \mathfrak{m}$ and $r > 0$.*

Proof. Note when $k = l$ that $\tilde{\rho} = 0$ if $(|1|,|2|) = (\bar{0},\bar{1})$ and $\tilde{\rho} = \varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$ if $(|1|,|2|) = (\bar{1},\bar{0})$. As noted above, it does no harm to change the choice of $\tilde{\rho}$ to assume in fact that $\tilde{\rho} = 0$ in both cases. Now we proceed to mimic the argument in [BK1, §12].

Consider the tensor algebra $T(M_l)$ in the (purely even) vector space M_l of $l \times l$ matrices over $\mathbb{C}$. For $1 \le i,j \le 2$, define a linear map $t_{i,j} : T(M_l) \to U(\mathfrak{g})$ by setting

$$t_{i,j}(1) := \delta_{i,j}, \qquad t_{i,j}(e_{a,b}) := (-1)^{|i|} e_{i*a,j*b},$$
$$t_{i,j}(x_1 \otimes \cdots \otimes x_r) := \sum_{1 \le h_1,\dots,h_{r-1} \le 2} t_{i,h_1}(x_1) t_{h_1,h_2}(x_2) \cdots t_{h_{r-1},j}(x_r),$$

for $1 \le a,b \le l$, $r \ge 1$ and $x_1,\dots,x_r \in M_l$, where $i * a$ denotes a if $|i| = \bar{0}$ and $l + a$ if $|i| = \bar{1}$. It is straightforward to check for $x, y_1, \dots, y_r \in M_l$ that

$$\begin{aligned}
&[t_{i,j}(x), t_{p,q}(y_1 \otimes \cdots \otimes y_r)] = \\
&\qquad (-1)^{|i||j|+|i||p|+|j||p|} \sum_{s=1}^{r} \big(t_{p,j}(y_1 \otimes \cdots \otimes y_{s-1}) t_{i,q}(x y_s \otimes \cdots \otimes y_r) \\
&\qquad\qquad - t_{p,j}(y_1 \otimes \cdots \otimes y_s x) t_{i,q}(y_{s+1} \otimes \cdots \otimes y_r) \big),
\end{aligned} \tag{5.2}$$

where the products xy_s and $y_s x$ on the right are ordinary matrix products in M_l. We extend $t_{i,j}$ to a $\mathbb{C}[u]$-module homomorphism $T(M_l)[u] \to U(\mathfrak{g})[u]$ in the obvious way. Introduce the following matrix with entries in the algebra $T(M_l)[u]$:

$$A(u) := \begin{pmatrix} u + e_{1,1} & e_{1,2} & e_{1,3} & \cdots & e_{1,l} \\ 1 & u + e_{2,2} & & & \vdots \\ 0 & & \ddots & & e_{l-2,l} \\ \vdots & & 1 & u + e_{l-1,l-1} & e_{l-1,l} \\ 0 & \cdots & 0 & 1 & u + e_{l,l} \end{pmatrix}$$

The point is that $t_{i,j;0}(u) = u^{-l}t_{i,j}(\mathrm{cdet}\, A(u))$, where the *column determinant* of an $l \times l$ matrix $A = (a_{i,j})$ with entries in a non-commutative ring means the Laplace expansion keeping all the monomials in column order, i.e. $\mathrm{cdet}\, A := \sum_{w \in S_l} \mathrm{sgn}(w) a_{w(1),1} \cdots a_{w(l),l}$. We also write $A_{c,d}(u)$ for the submatrix of $A(u)$ consisting only of rows and columns numbered $c, \dots, d$.

Since $\mathfrak{m}$ is generated by elements of the form $t_{i,j}(e_{c+1,c})$, it suffices now to show that $[t_{i,j}(e_{c+1,c}), t_{p,q}(\mathrm{cdet}\, A(u))]) \in \mathfrak{m}_\chi U(\mathfrak{g})$ for every $1 \leq i,j,p,q \leq 2$ and $c = 1, \dots, l-1$. To do this, we compute using the identity (5.2):

$$
\begin{aligned}
&[t_{i,j}(e_{c+1,c}), t_{p,q}(\mathrm{cdet}\, A(u))] = \\
&\qquad t_{p,j}(\mathrm{cdet}\, A_{1,c-1}(u)) t_{i,q}\left(\mathrm{cdet}\begin{pmatrix} e_{c+1,c} & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u+e_{c+1,c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u+e_{l,l} \end{pmatrix}\right) \\
&\qquad - t_{p,j}\left(\mathrm{cdet}\begin{pmatrix} u+e_{1,1} & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u+e_{c,c} & e_{c,c} \\ 0 & \cdots & 1 & e_{c+1,c} \end{pmatrix}\right) t_{i,q}(\mathrm{cdet}\, A_{c+2,l}(u)).
\end{aligned}
$$

In order to simplify the second term on right hand side, we observe crucially for $h = 1,2$ that $t_{h,j}\left((u+e_{c,c})\, e_{c+1,c}\right) \equiv t_{h,j}\left(u+e_{c,c}\right) \pmod{\mathfrak{m}_\chi U(\mathfrak{g})}$. Hence, we get that

$$
\begin{aligned}
&[t_{i,j}(e_{c+1,c}), t_{p,q}(\mathrm{cdet}\, A(u))] \equiv \\
&\qquad t_{p,j}(\mathrm{cdet}\, A_{1,c-1}(u)) t_{i,q}\left(\mathrm{cdet}\begin{pmatrix} 1 & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u+e_{c+1,c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u+e_{l,l} \end{pmatrix}\right) \\
&\qquad - t_{p,j}\left(\mathrm{cdet}\begin{pmatrix} u+e_{1,1} & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u+e_{c,c} & e_{c,c} \\ 0 & \cdots & 1 & 1 \end{pmatrix}\right) t_{i,q}(\mathrm{cdet}\, A_{c+2,l}(u))
\end{aligned}
$$

modulo $\mathfrak{m}_\chi U(\mathfrak{g})$. Making the obvious row and column operations gives that

$$
\mathrm{cdet}\begin{pmatrix} 1 & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u+e_{c+1,c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u+e_{l,l} \end{pmatrix} = u\, \mathrm{cdet}\, A_{c+2,l}(u),
$$

$$
\mathrm{cdet}\begin{pmatrix} u+e_{1,1} & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u+e_{c,c} & e_{c,c} \\ 0 & \cdots & 1 & 1 \end{pmatrix} = u\, \mathrm{cdet}\, A_{1,c-1}(u).
$$

It remains to substitute these into the preceeding formula. $\square$

We are ready to prove Proposition 4.4. Our argument goes by induction on $s_{2,1}+s_{1,2}=l-k$. For the base case $k=l$, we use Proposition 4.3 to rewrite the elements of Ω in terms of the elements $t_{i,j;0}^{(r)}$. The latter lie in W_π by Lemma 5.1. Hence so do the former.

Now assume that $s_{2,1}+s_{1,2}>0$. There are two cases according to to whether $s_{1,2}\geq s_{2,1}$ or $s_{2,1}>s_{1,2}$. Suppose first that $s_{1,2}\geq s_{2,1}$, hence that $s_{1,2}>0$. We may as well assume in addition that $l\geq 2$: the result is trivial for $l\leq 1$ as $\mathfrak{m}=\{0\}$. Let $\dot\pi$ be the pyramid obtained from π by removing the rightmost column. We will decorate all notation related to $\dot\pi$ with a dot to avoid any confusion. In particular $W_{\dot\pi}$ is a subalgebra of $U(\dot{\mathfrak{p}})\subseteq U(\dot{\mathfrak{g}})$. Let

$$\theta: U(\dot{\mathfrak{g}})\hookrightarrow U(\mathfrak{g})$$

be the embedding sending $e_{i,j}\in\dot{\mathfrak{g}}$ to $e_{i',j'}\in\mathfrak{g}$ if the ith and jth boxes of $\dot\pi$ correspond to the i'th and j'th boxes of π, respectively. Let b be the label of the box at the end of the second row of π, i.e. the box that gets removed when passing from π to $\dot\pi$. Also in the case that $s_{1,2}=1$ let c be the label of the box at the end of the first row of π.

Lemma 5.2. *In the above notation, the following hold:*

(i) $t_{1,1;0}^{(r)}=\theta(\dot t_{1,1;0}^{(r)})$ *for all* $r>0$*;*
(ii) $t_{2,1;1}^{(r)}=\theta(\dot t_{2,1;1}^{(r)})$ *for all* $r>s_{2,1}$*;*
(iii) $t_{1,2;1}^{(r)}=\theta(\dot t_{1,2;1}^{(r)})+\theta(\dot t_{1,2;1}^{(r-1)})S_{\tilde\rho}(\bar e_{b,b})-\left[\theta(\dot t_{1,2;1}^{(r-1)}),e_{b-1,b}\right]$ *for all* $r>s_{1,2}$*;*
(iv) $t_{2,2;1}^{(r)}=\theta(\dot t_{2,2;1}^{(r)})+\theta(\dot t_{2,2;1}^{(r-1)})S_{\tilde\rho}(\bar e_{b,b})-\left[\theta(\dot t_{2,2;1}^{(r-1)}),e_{b-1,b}\right]$ *for all* $r>0$.

Proof. This follows directly from the definition of these elements, using also that $\theta\circ S_{\dot{\tilde\rho}}=S_{\tilde\rho}\circ\theta$ on elements of $U(\dot{\mathfrak{p}})$. $\square$

Observe next that $\mathfrak{m}$ is generated by $\theta(\dot{\mathfrak{m}})\cup J$ where

$$J:=\begin{cases}\{e_{b,c},e_{b,b-1}\} & \text{if } s_{1,2}=1,\\ \{e_{b,b-1}\} & \text{if } s_{1,2}>1.\end{cases}\tag{5.3}$$

We know by induction that the following elements of $U(\dot{\mathfrak{p}})$ belong to $W_{\dot\pi}$: all $\dot t_{1,1;0}^{(r)}$ and $\dot t_{2,2;1}^{(r)}$ for $r\geq 0$; all $\dot t_{1,2;1}^{(r)}$ for $r\geq s_{1,2}$; all $\dot t_{2,1;1}^{(r)}$ for $r>s_{2,1}$. Also note that the elements of $\theta(\dot{\mathfrak{m}})$ commute with $e_{b-1,b}$ and $S_{\tilde\rho}(\bar e_{b,b})$. Combined with Lemma 5.2, we deduce that $[\theta(x),u]\in\theta(\dot{\mathfrak{m}}_\chi)U(\mathfrak{g})\subseteq\mathfrak{m}_\chi U(\mathfrak{g})$ for any $x\in\dot{\mathfrak{m}}$ and $u\in\Omega$. It remains to show that $[x,u]\in\mathfrak{m}_\chi U(\mathfrak{g})$ for each $x\in J$ and $u\in\Omega$. This is done in Lemmas 5.3, 5.4 and 5.6 below.

Lemma 5.3. *For* $x\in J$ *and* $u\in\{t_{1,1;0}^{(r)}\mid r>0\}\cup\{t_{2,1;1}^{(r)}\mid r>s_{2,1}\}$*, we have that* $[x,u]\in\mathfrak{m}_\chi U(\mathfrak{g})$.

Proof. Take $e_{b,d}\in J$. Consider a monomial $S_{\tilde\rho}(\bar e_{i_1,j_1}\cdots\bar e_{i_s,j_s})$ in the expansion of u from (4.10). The only way it could fail to supercommute with $e_{b,d}$ is if it involves some $\bar e_{i_h,j_h}$ with $j_h=b$ or $i_h=d$. Since $\mathrm{row}(j_s)=1$ and $\mathrm{col}(i_{h+1})>\mathrm{col}(j_h)$ when $\mathrm{row}(j_h)=2$, this situation arises only if $s_{1,2}=1$, $i_h=d$ and $j_h=c$. Then the supercommutator $[e_{b,d}, \bar{e}_{i_h,j_h}]$ equals $\pm e_{b,c}$. It remains to repeat this argument to see that we can move the resulting $e_{b,c} \in \mathfrak{m}_\chi$ to the beginning. $\square$

It is harder to deal with the remaining elements $t_{1,2;1}^{(r)}$ and $t_{2,2;1}^{(r)}$ of Ω. We follow different approaches according to whether $s_{1,2} > 1$ or $s_{1,2} = 1$.

Lemma 5.4. *Assume that $s_{1,2} > 1$. We have that $[e_{b,b-1}, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $u \in \{t_{1,2;1}^{(r)} \mid r > s_{1,2}\} \cup \{t_{2,2;1}^{(r)} \mid r > 0\}$.*

Proof. We just explain in detail for $u = t_{1,2;1}^{(r)}$; the other case follows the same pattern. Let $\ddot{\pi}$ be the pyramid obtained from π by removing its rightmost two columns. We decorate all notation associated to $W_{\ddot{\pi}}$ with a double dot, so $W_{\ddot{\pi}} \subseteq U(\ddot{\mathfrak{p}}) \subseteq U(\ddot{\mathfrak{g}})$ and so on. Let

$$\phi : U(\ddot{\mathfrak{g}}) \hookrightarrow U(\mathfrak{g})$$

be the embedding sending $e_{i,j} \in \ddot{\mathfrak{g}}$ to $e_{i',j'} \in \mathfrak{g}$ where the ith and jth boxes of $\ddot{\pi}$ are labelled by i and j in π, respectively. For $r \geq s_{1,2}$, we have by analogy with Lemma 5.2(iii) that

$$\theta(\dot{t}_{1,2;1}^{(r)}) = \phi(\ddot{t}_{1,2;1}^{(r)}) + \phi(\ddot{t}_{1,2;1}^{(r-1)}) S_{\tilde{\rho}}(\bar{e}_{b-1,b-1}) - \big[\phi(\ddot{t}_{1,2;1}^{(r-1)}), e_{b-2,b-1}\big].$$

We combine this with Lemma 5.2(iii) to deduce for $r > s_{1,2}$ that

$$\begin{aligned} t_{1,2;1}^{(r)} =\, & \phi(\ddot{t}_{1,2;1}^{(r)}) + \phi(\ddot{t}_{1,2;1}^{(r-1)}) S_{\tilde{\rho}}(\bar{e}_{b-1,b-1}) - \big[\phi(\ddot{t}_{1,2;1}^{(r-1)}), e_{b-2,b-1}\big] \\ & + \phi(\ddot{t}_{1,2;1}^{(r-1)}) S_{\tilde{\rho}}(\bar{e}_{b,b}) + \phi(\ddot{t}_{1,2;1}^{(r-2)}) S_{\tilde{\rho}}(\bar{e}_{b-1,b-1}) S_{\tilde{\rho}}(\bar{e}_{b,b}) \\ & - \big[\phi(\ddot{t}_{1,2;1}^{(r-2)}), e_{b-2,b-1}\big] S_{\tilde{\rho}}(\bar{e}_{b,b}) - \phi(\ddot{t}_{1,2;1}^{(r-2)}) \bar{e}_{b-1,b} + \big[\phi(\ddot{t}_{1,2;1}^{(r-2)}), e_{b-2,b}\big]. \end{aligned}$$

We deduce that

$$\begin{aligned} [e_{b,b-1}, t_{1,2;1}^{(r)}] =\, & \phi(\ddot{t}_{1,2;1}^{(r-2)}) \big(\bar{e}_{b,b-1} S_{\tilde{\rho}}(\bar{e}_{b,b}) - \bar{e}_{b,b-1} S_{\tilde{\rho}}(\bar{e}_{b-1,b-1}) + (-1)^{|2|} \bar{e}_{b,b-1}\big) \\ & + \big[\phi(\ddot{t}_{1,2;1}^{(r-2)}), e_{b-2,b-1}\big] \bar{e}_{b,b-1} - \phi(\ddot{t}_{1,2;1}^{(r-2)}) (\bar{e}_{b,b} - \bar{e}_{b-1,b-1}) \\ & - \big[\phi(\ddot{t}_{1,2;1}^{(r-2)}), e_{b-2,b-1}\big]. \end{aligned}$$

Working modulo $\mathfrak{m}_\chi U(\mathfrak{g})$, we can replace all $\bar{e}_{b,b-1}$ by 1. Then we are reduced just to checking that

$$S_{\tilde{\rho}}(\bar{e}_{b,b}) - S_{\tilde{\rho}}(\bar{e}_{b-1,b-1}) + (-1)^{|2|} = \bar{e}_{b,b} - \bar{e}_{b-1,b-1}.$$

This follows because $(\tilde{\rho}|\varepsilon_b) - (\tilde{\rho}|\varepsilon_{b-1}) + (-1)^{|2|} = 0$ by the definition (4.8). $\square$

Lemma 5.5. *Assume that $s_{1,2} = 1$. For $r > 2$ we have that*

$$t_{1,2;1}^{(r)} = (-1)^{|1|} \big[t_{1,1;0}^{(2)}, t_{1,2;1}^{(r-1)}\big] - t_{1,1;0}^{(1)} t_{1,2;1}^{(r-1)}, \tag{5.4}$$

$$t_{2,2;1}^{(r)} = (-1)^{|1|} \big[t_{1,2;1}^{(2)}, t_{2,1;1}^{(r-1)}\big] - \sum_{a=0}^{r} t_{1,1;1}^{(a)} t_{2,2;1}^{(r-a)}. \tag{5.5}$$

Proof. We prove (5.4). The induction hypothesis means that we can appeal to Theorem 4.5 for the algebra $W_{\dot{\pi}}$. Hence using the relations from Theorem 2.2, we know that the following hold in the algebra $W_{\dot{\pi}}$ for all $r \geq 2$:

$$\dot{t}_{1,2;1}^{(r)} = (-1)^{|1|} \big[\dot{t}_{1,1;0}^{(2)}, \dot{t}_{1,2;1}^{(r-1)}\big] - \dot{t}_{1,1;0}^{(1)} \dot{t}_{1,2;1}^{(r-1)}.$$

Using Lemma 5.2, we deduce for $r > 2$ that

$$\begin{aligned}
t_{1,2;1}^{(r)} &= \theta(\dot{t}_{1,2;1}^{(r)}) + \theta(\dot{t}_{1,2;1}^{(r-1)})S_{\tilde{\rho}}(\bar{e}_{b,b}) - \big[\theta(\dot{t}_{1,2;1}^{(r-1)}), e_{b-1,b}\big] \\
&= (-1)^{|1|}\big[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-1)})\big] - t_{1,1;0}^{(1)}\theta(\dot{t}_{1,2;1}^{(r-1)}) \\
&\quad + (-1)^{|1|}\big[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-2)})\big]S_{\tilde{\rho}}(\bar{e}_{b,b}) - t_{1,1;0}^{(1)}\theta(\dot{t}_{1,2;1}^{(r-2)})S_{\tilde{\rho}}(\bar{e}_{b,b}) \\
&\quad - (-1)^{|1|}\Big[\big[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-2)})\big], e_{b-1,b}\Big] + \big[t_{1,1;0}^{(1)}\theta(\dot{t}_{1,2;1}^{(r-2)}), e_{b-1,b}\big] \\
&= (-1)^{|1|}\Big[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-1)}) + \theta(\dot{t}_{1,2;1}^{(r-2)})S_{\tilde{\rho}}(\bar{e}_{b,b}) - \big[\theta(\dot{t}_{1,2;1}^{(r-2)}), e_{b-1,b}\big]\Big] \\
&\quad - t_{1,1;0}^{(1)}\Big(\theta(\dot{t}_{1,2;1}^{(r-1)}) + \theta(\dot{t}_{1,2;1}^{(r-2)})S_{\tilde{\rho}}(\bar{e}_{b,b}) - \big[\theta(\dot{t}_{1,2;1}^{(r-2)}), e_{b-1,b}\big]\Big) \\
&= (-1)^{|1|}\big[t_{1,1;0}^{(2)}, t_{1,2;1}^{(r-1)}\big] - t_{1,1;0}^{(1)}t_{1,2;1}^{(r-1)}.
\end{aligned}$$

The other equation (5.5) follows by a similar trick. $\square$

Lemma 5.6. *Assume that $s_{1,2} = 1$. We have that $[x, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $x \in J$ and $u \in \{t_{1,2;1}^{(r)} \mid r > s_{1,2}\} \cup \{t_{2,2;1}^{(r)} \mid r > 0\}$.*

Proof. Proceed by induction on r. The base cases when $r \leq 2$ are small enough that they can be checked directly from the definitions. Then for $r > 2$ use Lemma 5.5, noting by the induction hypothesis and Lemma 5.3 that all the terms on the right hand side of (5.4)–(5.5) are already known to lie in $\mathfrak{m}_\chi U(\mathfrak{g})$. $\square$

We have now verified the induction step in the case that $s_{1,2} \geq s_{2,1}$. It remains to establish the induction step when $s_{2,1} > s_{1,2}$. The strategy for this is sufficiently similar to the case just done (based on removing columns from the left of the pyramid π) that we leave the details to the reader. We just note one minor difference: in the proof of the analogue of Lemma 5.2 it is no longer the case that $\theta \circ S_{\dot{\tilde{\rho}}} = S_{\tilde{\rho}} \circ \theta$, but this can be fixed by allowing the choice of $\tilde{\rho}$ to change by a multiple of $\varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$.

This completes the proof of Proposition 4.4.

6. Triangular decomposition

Let W_π be the principal W-algebra in $\mathfrak{g} = \mathfrak{gl}_{m|n}(\mathbb{C})$ associated to pyramid π. We adopt all the notation from §4. So:

- $(|1|, |2|)$ is a parity sequence chosen so that $(|1|, |2|) = (\bar{0}, \bar{1})$ if $m < n$ and $(|1|, |2|) = (\bar{1}, \bar{0})$ if $m > n$;
- π has $k = \min(m, n)$ boxes in its first row and $l = \max(m, n)$ boxes in its second row;
- $\sigma = (s_{i,j})_{1 \leq i,j \leq 2}$ is a shift matrix compatible with π.

We identify W_π with Y_σ^l, the shifted Yangian of level l, via the isomorphism μ from (4.18). Thus we have available a set of Drinfeld generators for W_π satisfying the relations from Theorem 2.2 plus the additional truncation relations $d_1^{(r)} = 0$ for $r > k$. In view of (4.19)–(4.21) and (4.10), we even have available explicit formulae for these generators as elements of $U(\mathfrak{p})$, although we seldom need to use these (but see the proof of Lemma 8.3 below).

By the relations, W_π admits a $\mathbb{Z}$-grading

$$W_\pi = \bigoplus_{g \in \mathbb{Z}} W_{\pi;g}$$

such that the generators $d_i^{(r)}$ are of degree 0, the generators $e^{(r)}$ are of degree 1, and the generators $f^{(r)}$ are of degree -1. Moreover the PBW theorem (Corollary 3.6) implies that $W_{\pi;g} = 0$ for $|g| > k$.

More surprisingly, the algebra W_π admits a triangular decomposition. To introduce this, let W_π^0 (resp. W_π^+, resp. W_π^-) be the subalgebra of W_π generated by the elements $\Omega_0 := \{d_1^{(r)}, d_2^{(s)} \mid 0 < r \leq k, 0 < s \leq l\}$ (resp. $\Omega_+ := \{e^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\}$, resp. $\Omega_- := \{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}$). Let $W_\pi^\sharp$ (resp. $W_\pi^\flat$) be the subalgebra of W_π generated by $\Omega_0 \cup \Omega_+$ (resp. by $\Omega_- \cup \Omega_0$). We warn the reader that the elements $e^{(r)}$ $(r > s_{1,2} + k)$ do not necessarily lie in W_π^+ (but they do lie in $W_\pi^\sharp$ by (3.5)). Similarly the elements $f^{(r)}$ $(r > s_{2,1} + k)$ do not necessarily lie in W_π^- (but they do lie in $W_\pi^\flat$), and the elements $d_2^{(r)}$ $(r > l)$ do not necessarily lie in any of $W_\pi^0, W_\pi^\sharp$ or $W_\pi^\flat$.

Theorem 6.1. *The algebras W_π^0, W_π^+ and W_π^- are free supercommutative superalgebras on generators Ω_0, Ω_+ and Ω_-, respectively. Multiplication defines vector space isomorphisms*

$$W_\pi^- \otimes W_\pi^0 \otimes W_\pi^+ \xrightarrow{\sim} W_\pi,$$
$$W_\pi^0 \otimes W_\pi^+ \xrightarrow{\sim} W_\pi^\sharp, \qquad W_\pi^- \otimes W_\pi^0 \xrightarrow{\sim} W_\pi^\flat.$$

Moreover, there are unique surjective homomorphisms

$$W_\pi^\sharp \twoheadrightarrow W_\pi^0, \qquad W_\pi^\flat \twoheadrightarrow W_\pi^0$$

sending $e^{(r)} \mapsto 0$ for all $r > s_{1,2}$ or $f^{(r)} \mapsto 0$ for all $r > s_{2,1}$, respectively, such that the restriction of these maps to the subalgebra W_π^0 is the identity.

Proof. Throughout the proof, we repeatedly apply the PBW theorem (Corollary 3.6), choosing the order of generators so that $\Omega_- < \Omega_0 < \Omega_+$.

To start with, note by the left hand relations in Theorem 2.2 that each of W_π^0, W_π^+ and W_π^- is supercommutative. Combined with the PBW theorem, we deduce that they are free supercommutative on the given generators. Moreover the PBW theorem implies that the multiplication map $W_\pi^- \otimes W_\pi^0 \otimes W_\pi^+ \to W_\pi$ is a vector space isomorphism.

Next we observe that $W_\pi^\sharp$ contains all the elements $e^{(r)}$ $(r > s_{1,2})$. This follows from (3.5) by induction on r. Moreover it is spanned as a vector space by the ordered supermonomials in the generators $\Omega_0 \cup \Omega_+$. This follows from (3.5), the relation for $[d_i^{(r)}, e^{(s)}]$ in Theorem 2.2, and induction on Kazhdan degree. Hence the multiplication map $W_\pi^0 \otimes W_\pi^+ \to W_\pi^\sharp$ is surjective. It is injective by the PBW theorem, so it is an isomorphism. Similarly $W_\pi^- \otimes W_\pi^0 \to W_\pi^\flat$ is an isomorphism.

Finally, let $J^\sharp$ be the two-sided ideal of $W_\pi^\sharp$ that is the sum of all of the graded components $W_{\pi;g}^\sharp := W_\pi^\sharp \cap W_{\pi;g}$ for $g > 0$. By the PBW theorem, the natural quotient map $W_\pi^0 \to W_\pi^\sharp / J^\sharp$ is an isomorphism. Hence there is a surjection

$W_\pi^\sharp \twoheadrightarrow W_\pi^0$ as in the statement of the theorem. A similar argument yields the desired surjection $W_\pi^\flat \twoheadrightarrow W_\pi^0$. □

7. Irreducible representations

Continue with the notation of the previous section. Using the triangular decomposition, we can classify irreducible W_π-modules by highest weight theory. Define a *π-tableau* to be a filling of the boxes of the pyramid π by arbitrary complex numbers. Let Tab_π denote the set of all such π-tableaux. We represent the π-tableau with entries $a_1, \dots, a_k$ along its first row and $b_1, \dots, b_l$ along its second row simply by the array ${}^{a_1 \cdots a_k}_{b_1 \cdots b_l}$. We say that $A, B \in \mathrm{Tab}_\pi$ are *row equivalent*, denoted $A \sim B$, if B can be obtained from A by permuting entries within each row.

Recall from Theorem 6.1 that W_π^0 is the polynomial algebra on $\{d_1^{(r)}, d_2^{(s)} \mid 0 < r \le k, 0 < s \le l\}$. For $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l} \in \mathrm{Tab}_\pi$, let $\mathbb{C}_A$ be the one-dimensional W_π^0-module on basis 1_A such that

$$u^k d_1(u) 1_A = (u + a_1) \cdots (u + a_k) 1_A, \tag{7.1}$$

$$u^l d_2(u) 1_A = (u + b_1) \cdots (u + b_l) 1_A. \tag{7.2}$$

Thus $d_1^{(r)} 1_A = e_r(a_1, \dots, a_k) 1_A$ and $d_2^{(r)} 1_A = e_r(b_1, \dots, b_l) 1_A$, where e_r denotes the rth elementary symmetric polynomial. Every irreducible W_π^0-module is isomorphic to $\mathbb{C}_A$ for some $A \in \mathrm{Tab}_\pi$, and $\mathbb{C}_A \cong \mathbb{C}_B$ if and only if $A \sim B$.

Given $A \in \mathrm{Tab}_\pi$, we view $\mathbb{C}_A$ as a $W_\pi^\sharp$-module via the surjection $W_\pi^\sharp \twoheadrightarrow W_\pi^0$ from Theorem 6.1, i.e. $e^{(r)} 1_A = 0$ for all $r > s_{1,2}$. Then we induce to form the *Verma module*

$$\overline{M}(A) := W_\pi \otimes_{W_\pi^\sharp} \mathbb{C}_A. \tag{7.3}$$

Sometimes we need to view this as a supermodule, which we do by declaring that its cyclic generator $1 \otimes 1_A$ is even. By Theorem 6.1, W_π is a free right $W_\pi^\sharp$-module with basis given by the ordered supermonomials in the odd elements $\{f^{(r)} \mid s_{2,1} < r \le s_{2,1} + k\}$. Hence $\overline{M}(A)$ has basis given by the vectors $x \otimes 1_A$ as x runs over this set of supermonomials. In particular $\dim \overline{M}(A) = 2^k$.

The following lemma shows that $\overline{M}(A)$ has a unique irreducible quotient which we denote by $\overline{L}(A)$; we write v_+ for the image of $1 \otimes 1_A \in \overline{M}(A)$ in $\overline{L}(A)$.

Lemma 7.1. *For $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l} \in \mathrm{Tab}_\pi$, the Verma module $\overline{M}(A)$ has a unique irreducible quotient $\overline{L}(A)$. The image v_+ of $1 \otimes 1_A$ is the unique (up to scalars) non-zero vector in $\overline{L}(A)$ such that $e^{(r)} v_+ = 0$ for all $r > s_{1,2}$. Moreover we have that $d_1^{(r)} v_+ = e_r(a_1, \dots, a_k) v_+$ and $d_2^{(r)} v_+ = e_r(b_1, \dots, b_l) v_+$ for all $r \ge 0$.*

Proof. Let $\lambda := (-1)^{|1|}(a_1 + \cdots + a_k)$. For any $\mu \in \mathbb{C}$, let $\overline{M}(A)_\mu$ be the μ-eigenspace of the endomorphism of $\overline{M}(A)$ defined by $d := (-1)^{|1|} d_1^{(1)} \in W_\pi$. Note by (7.1) and the relations that $d 1_A = \lambda 1_A$ and $[d, f^{(r)}] = -f^{(r)}$ for each $r > s_{2,1}$. Using the PBW basis for $\overline{M}(A)$, it follows that

$$\overline{M}(A) = \bigoplus_{i=0}^{k} \overline{M}(A)_{\lambda - i} \tag{7.4}$$

and $\dim \overline{M}(A)_{\lambda-i} = \binom{k}{i}$ for each $0 \leq i \leq k$. In particular, $\overline{M}(A)_\lambda$ is one-dimensional, and it generates $\overline{M}(A)$ as a $W_\pi^\flat$-module. This is all that is needed to deduce that $\overline{M}(A)$ has a unique irreducible quotient $\overline{L}(A)$ following the standard argument of highest weight theory.

The vector v_+ is a non-zero vector annihilated by $e^{(r)}$ $(r > s_{1,2})$, and $d_1^{(r)}v_+$ and $d_2^{(r)}v_+$ are as stated thanks to (7.1)–(7.2). It just remains to show that any vector $v \in \overline{L}(A)$ annihilated by all $e^{(r)}$ is a multiple of v_+. The decomposition (7.4) induces an analogous decomposition

$$\overline{L}(A) = \bigoplus_{i=0}^{k} \overline{L}(A)_{\lambda-i}, \tag{7.5}$$

although for $0 < i \leq k$ the eigenspace $\overline{L}(A)_{\lambda-i}$ may now be zero. Write $v = \sum_{i=0}^k v_i$ with $v_i \in \overline{L}(A)_{\lambda-i}$. Then we need to show that $v_i = 0$ for $i > 0$. We have that $e^{(r)}v = \sum_{i=1}^k e^{(r)}v_i = 0$, hence $e^{(r)}v_i = 0$ for each i. But this means for $i > 0$ that the submodule $W_\pi v_i = W_\pi^\flat v_i$ has trivial intersection with $\overline{L}(A)_\lambda$, hence it must be zero. □

Here is the classification of irreducible W_π-modules.

Theorem 7.2. *Every irreducible W_π-module is finite dimensional and is isomorphic to one of the modules $\overline{L}(A)$ from Lemma 7.1 for some $A \in \mathrm{Tab}_\pi$. Moreover $\overline{L}(A) \cong \overline{L}(B)$ if and only if $A \sim B$. Hence, fixing a set $\mathrm{Tab}_\pi /\!\sim$ of representatives for the $\sim$-equivalence classes in Tab_π, the modules*

$$\{\overline{L}(A) \mid A \in \mathrm{Tab}_\pi /\!\sim\}$$

give a complete set of pairwise inequivalent irreducible W_π-modules.

Proof. We note to start with for $A, B \in \mathrm{Tab}_\pi$ that $\overline{L}(A) \cong \overline{L}(B)$ if and only if $A \sim B$. This is clear from Lemma 7.1.

Now take an arbitrary (conceivably infinite dimensional) irreducible W_π-module L. We want to show that $L \cong \overline{L}(A)$ for some $A \in \mathrm{Tab}_\pi$. For $i \geq 0$, let

$$L[i] := \left\{ v \in L \;\middle|\; W_{\pi;g}v = \{0\} \text{ if } g > 0 \text{ or } g \leq -i \right\}.$$

We claim initially that $L[k+1] \neq \{0\}$. To see this, recall that $W_{\pi;g} = \{0\}$ for $g \leq -k-1$, so by the PBW theorem $L[k+1]$ is simply the set of all vectors $v \in L$ such that $e^{(r)}v = 0$ for all $s_{1,2} < r \leq s_{1,2} + k$. Now take any non-zero vector $v \in L$ such that $\#\{r = s_{1,2}+1, \dots, s_{1,2}+k \mid e^{(r)}v = 0\}$ is maximal. If $e^{(r)}v \neq 0$ for some $s_{1,2} < r \leq s_{1,2}+k$, we can replace v by $e^{(r)}v$ to get a non-zero vector annihilated by more $e^{(r)}$'s. Hence $v \in L[k+1]$ by the maximality of the choice of v, and we have shown that $L[k+1] \neq \{0\}$.

Since $L[k+1] \neq \{0\}$ it makes sense to define $i \geq 0$ to be minimal such that $L[i] \neq \{0\}$. Since $L[0] = \{0\}$, we actually have that $i > 0$. Pick $0 \neq v \in L[i]$ and let $L' := W_\pi^\sharp v$. Actually, by the PBW theorem, we have that $L' = W_\pi^0 v$, and $L' \subseteq L[i]$. Suppose first that L' is irreducible as a W_π^0-module. Then $L' \cong \mathbb{C}_A$ for some $A \in \mathrm{Tab}_\pi$. The inclusion $L' \hookrightarrow L$ induces a non-zero W_π-module homomorphism

$$\overline{M}(A) \cong W_\pi \otimes_{W_\pi^\sharp} L' \to L,$$

which is surjective as L is irreducible. Hence $L \cong \overline{L}(A)$.

It remains to rule out the possibility that L' is reducible. Suppose for a contradiction that L' possesses a non-zero proper W_π^0-submodule L''. As $L = W_\pi L''$ and $W_\pi^\sharp L'' = L''$, the PBW theorem implies that we can write

$$v = w + \sum_{h=1}^{k} \sum_{s_{2,1} < r_1 < \cdots < r_h \leq s_{2,1}+k} f^{(r_1)} \cdots f^{(r_h)} v_{r_1,\dots,r_h}$$

for some vectors $v_{r_1,\dots,r_h}, w \in L''$. Then we have that

$$0 \neq v - w \in L[i] \cap \left(\sum_{g \leq -1} W_{\pi;g} L[i] \right) \subseteq L[i-1].$$

This shows $L[i-1] \neq \{0\}$, contradicting the minimality of the choice of i. $\square$

The final theorem of the section gives an explicit monomial basis for $\overline{L}(A)$. We only prove linear independence here; the spanning part of the argument will be given in the next section.

Theorem 7.3. *Suppose $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l} \in \mathrm{Tab}_\pi$. Let $h \geq 0$ be maximal such that there exist distinct $1 \leq i_1, \dots, i_h \leq k$ and distinct $1 \leq j_1, \dots, j_h \leq l$ with $a_{i_1} = b_{j_1}, \dots, a_{i_h} = b_{j_h}$. Then the irreducible module $\overline{L}(A)$ has basis given by the vectors xv_+ as x runs over all ordered supermonomials in the odd elements $\{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k - h\}$.*

Proof. Let $\bar{k} := k-h$ and $\bar{l} := l-h$. Since $\overline{L}(A)$ only depends on the $\sim$-equivalence class of A, we can reindex to assume that $a_{\bar{k}+1} = b_{\bar{l}+1}, a_{\bar{k}+2} = b_{\bar{l}+2}, \dots, a_k = b_l$. We proceed to show that the vectors xv_+ for all ordered supermonomials x in $\{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + \bar{k}\}$ are linearly independent in $\overline{L}(A)$. In fact it is enough for this to show just that

$$f^{(s_{2,1}+1)} f^{(s_{2,1}+2)} \cdots f^{(s_{2,1}+\bar{k})} v_+ \neq 0. \tag{7.6}$$

Indeed, assuming (7.6), we can prove the linear independence in general by taking any non-trivial linear relation of the form

$$\sum_{a=0}^{\bar{k}} \sum_{s_{2,1} < r_1 < \cdots < r_a \leq s_{2,1}+\bar{k}} \lambda_{r_1,\dots,r_a} f^{(r_1)} \cdots f^{(r_a)} v_+ = 0.$$

Let a be minimal such that $\lambda_{r_1,\dots,r_a} \neq 0$ for some $r_1, \dots, r_a$. Apply $f^{(s_1)} \cdots f^{(s_{\bar{k}-a})}$ where $s_{2,1} < s_1 < \cdots < s_{\bar{k}-a} \leq s_{2,1} + \bar{k}$ are different from $r_1 < \cdots < r_a$. All but one term of the summation becomes zero and using (7.6) we can deduce that $\lambda_{r_1,\dots,r_a} = 0$, a contradiction.

In this paragraph, we prove (7.6) by showing that

$$e^{(s_{1,2}+1)} e^{(s_{1,2}+2)} \cdots e^{(s_{1,2}+\bar{k})} f^{(s_{2,1}+1)} f^{(s_{2,1}+2)} \cdots f^{(s_{2,1}+\bar{k})} v_+ \neq 0. \tag{7.7}$$

The left hand side of (7.7) equals

$$\sum_{w \in S_{\bar{k}}} \mathrm{sgn}(w) \left[e^{(\bar{k}+1+s_{1,2}-1)}, f^{(s_{2,1}+w(1))} \right] \cdots \left[e^{(\bar{k}+1+s_{1,2}-\bar{k})}, f^{(s_{2,1}+w(\bar{k}))} \right] v_+.$$

By Remark 2.3, up to a sign, this is $\det\left(c^{(\bar{l}-i+j)}\right)_{1\le i,j\le \bar{k}} v_+$. It is easy to see from Lemma 7.1 that $c^{(r)}v_+ = e_r(b_1,\dots,b_{\bar{l}}/a_1,\dots,a_{\bar{k}})v_+$ where

$$e_r(b_1,\dots,b_{\bar{l}}/a_1,\dots,a_{\bar{k}}) := \sum_{s+t=r}(-1)^t e_s(b_1,\dots,b_{\bar{l}})h_t(a_1,\dots,a_{\bar{k}})$$

is the rth elementary supersymmetric function from [M, Exercise I.3.23]. Thus we need to show that $\det\left(e_{\bar{l}-i+j}(b_1,\dots,b_{\bar{l}}/a_1,\dots,a_{\bar{k}})\right)_{1\le i,j\le\bar{k}} \neq 0$. But this determinant is exactly the supersymmetric Schur function $s_\lambda(b_1,\dots,b_{\bar{l}}/a_1,\dots,a_{\bar{k}})$ defined in [M, Exercise I.3.23] for the partition $\lambda = (\bar{k}^{\bar{l}})$. Hence by the factorization property described there, it is equal to $\prod_{1\le i\le\bar{l}}\prod_{1\le j\le\bar{k}}(b_i - a_j)$, which is indeed non-zero.

We have now proved the linear independence of the vectors xv_+ as x runs over all ordered supermonomials in $\{f^{(r)} \mid s_{2,1} < r \le s_{2,1}+\bar{k}\}$. It remains to show that these vectors also span $\overline{L}(A)$. For this, it is enough to show that $\dim \overline{L}(A) \le 2^{\bar{k}}$. This will be established in the next section by means of an explicit construction of a module of dimension $2^{\bar{k}}$ containing $\overline{L}(A)$ as a subquotient. □

8. Tensor products

In this section we define some more general comultiplications between the algebras W_π, allowing certain tensor products to be defined. We apply this to construct so-called *standard modules* $\overline{V}(A)$ for each $A \in \mathrm{Tab}_\pi$. Then we complete the proof of Theorem 7.3 by showing that every irreducible W_π-module is isomorphic to one of the modules $\overline{V}(A)$ for suitable A.

Recall that the pyramid π has l boxes on its second row. Suppose we are given $l_1,\dots,l_d \ge 0$ such that $l_1+\dots+l_d = l$. For each $c = 1,\dots,d$, let π_c be the pyramid consisting of columns $l_1+\dots+l_{c-1}+1,\dots,l_1+\dots+l_c$ of π. Thus π is the "concatenation" of the pyramids $\pi_1,\dots,\pi_d$. Let W_{π_c} be the principal W-algebra defined from π_c. Let $\sigma_1,\dots,\sigma_d$ be the unique shift matrices such that each σ_c is compatible with π_c, and σ_c is lower (resp. upper) triangular if $s_{2,1} \ge l_1+\dots+l_c$ (resp. $s_{1,2} \ge l_c+\dots+l_d$). We denote the Miura transform for W_{π_c} by $\mu_c : W_{\pi_c} \hookrightarrow U_{\sigma_c}^{l_c}$.

Lemma 8.1. *With the above notation, there is a unique injective algebra homomorphism*

$$\Delta_{l_1,\dots,l_d} : W_\pi \hookrightarrow W_{\pi_1}\otimes\dots\otimes W_{\pi_d} \tag{8.1}$$

such that $(\mu_1\otimes\dots\otimes\mu_d)\circ\Delta_{l_1,\dots,l_d} = \mu$.

Proof. Let us add the suffix c to all notation arising from the definition of W_{π_c}, so that W_{π_c} is a subalgebra of $U(\mathfrak{p}_c)$, we have that $\mathfrak{g}_c = \mathfrak{m}_c \oplus \mathfrak{h}_c \oplus \mathfrak{p}_c^\perp$, and so on. We identify $\mathfrak{g}_1\oplus\dots\oplus\mathfrak{g}_d$ with a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ so that $e_{i,j}\in\mathfrak{g}_c$ is identified with $e_{i',j'}\in\mathfrak{g}$ where i' and j' are the labels of the boxes of π corresponding to the ith and jth boxes of π_c, respectively. Similarly we identify $\mathfrak{m}_1\oplus\dots\oplus\mathfrak{m}_d$ with $\mathfrak{m}'\subseteq\mathfrak{m}$, $\mathfrak{p}_1\oplus\dots\oplus\mathfrak{p}_d$ with $\mathfrak{p}'\subseteq\mathfrak{p}$, and $\mathfrak{h}_1\oplus\dots\oplus\mathfrak{h}_d$ with $\mathfrak{h}'=\mathfrak{h}$. Also let $\tilde\rho' := \tilde\rho_1+\dots+\tilde\rho_d$, a character of $\mathfrak{p}'$. In this way $W_{\pi_1}\otimes\dots\otimes W_{\pi_d}$ is identified with $W'_\pi := \{u\in U(\mathfrak{p}') \mid u\mathfrak{m}'_\chi \subseteq \mathfrak{m}'_\chi U(\mathfrak{g}')\}$, where $\mathfrak{m}'_\chi = \{x-\chi(x) \mid x\in\mathfrak{m}'\}$.

Let $\mathfrak{q}$ be the unique parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}'$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Let $\psi : U(\mathfrak{q}) \twoheadrightarrow U(\mathfrak{g}')$ be the homomorphism induced by the natural projection of $\mathfrak{q} \twoheadrightarrow \mathfrak{g}'$. The following diagram commutes:

$$\begin{tikzcd} U(\mathfrak{p}) \arrow[r, "S_{-\tilde\rho'}\circ\psi\circ S_{\tilde\rho}"] \arrow[d, "\mathrm{pr}\circ S_{\tilde\rho}"'] & U(\mathfrak{p}') \arrow[d, "\mathrm{pr}'\circ S_{\tilde\rho'}"] \\ U(\mathfrak{h}) \arrow[r, equal] & U(\mathfrak{h}') \end{tikzcd}$$

We claim that $S_{-\tilde\rho'} \circ \psi \circ S_{\tilde\rho}$ maps W_π into W'_π. The claim implies the lemma, for then it makes sense to *define* $\Delta_{l_1,\dots,l_d}$ to be the restriction of this map to W_π, and we are done by the commutativity of the above diagram and injectivity of the Miura transform.

To prove the claim, observe that $\tilde\rho - \tilde\rho'$ extends to a character of $\mathfrak{q}$, hence there is a corresponding shift automorphism $S_{\tilde\rho-\tilde\rho'} : U(\mathfrak{q}) \to U(\mathfrak{q})$ which preserves W'_π. Moreover $S_{-\tilde\rho'} \circ \psi \circ S_{\tilde\rho} = S_{\tilde\rho-\tilde\rho'} \circ \psi$. Therefore it enough to check just that $\psi(W_\pi) \subseteq W'_\pi$. To see this, take $u \in W_\pi$, so that $u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})$. This implies that $u\mathfrak{m}'_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g}) \cap U(\mathfrak{q})$, hence applying ψ we get that $\psi(u)\mathfrak{m}'_\chi \subseteq \mathfrak{m}'_\chi U(\mathfrak{g}')$. This shows that $\psi(u) \in W'_\pi$ as required. $\square$

Remark 8.2. Special cases of the maps (8.1) with $d = 2$ are related to the comultiplications Δ, Δ_+ and Δ_- from (2.14)–(2.16). Indeed, if $l = l_1+l_2$ for $l_1 \geq s_{2,1}$ and $l_2 \geq s_{1,2}$, the shift matrices σ_1 and σ_2 above are equal to σ^{lo} and σ^{up}, respectively. Both squares in the following diagram commute:

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta"] \arrow[d, "\mathrm{ev}^l_\sigma"'] & Y_{\sigma_1} \otimes Y_{\sigma_2} \arrow[d, "\mathrm{ev}^{l_1}_{\sigma_1}\otimes \mathrm{ev}^{l_2}_{\sigma_2}"] \\ U^l_\sigma \arrow[r, equal] & U^{l_1}_{\sigma_1} \otimes U^{l_2}_{\sigma_2} \\ W_\pi \arrow[r, "\Delta_{l_1,l_2}"] \arrow[u, "\mu"] & W_{\pi_1} \otimes W_{\pi_2} \arrow[u, "\mu_1\otimes\mu_2"'] \end{tikzcd}$$

Indeed, the top square commutes by the definition of the evaluation homomorphisms from (3.2), while the bottom square commutes by Lemma 8.1. Hence, under our isomorphism between principal W-algebras and truncated shifted Yangians, $\Delta_{l_1,l_2} : W_\pi \to W_{\pi_1} \otimes W_{\pi_2}$ corresponds exactly to the map $Y^l_\sigma \to Y^{l_1}_{\sigma_1} \otimes Y^{l_2}_{\sigma_2}$ induced by the comultiplication $\Delta : Y_\sigma \to Y_{\sigma_1} \otimes Y_{\sigma_2}$.

Instead, if $l_1 = l - 1$, $l_2 = 1$ and the rightmost column of π consists of a single box, the map $\Delta_{l-1,1} : W_\pi \to W_{\pi_1} \otimes U(\mathfrak{gl}_1)$ corresponds exactly to the map $Y^l_\sigma \to Y^{l-1}_{\sigma_+} \otimes U(\mathfrak{gl}_1)$ induced by $\Delta_+ : Y_\sigma \to Y_{\sigma_+} \otimes U(\mathfrak{gl}_1)$. Similarly, if $l_1 = 1, l_2 = l - 1$ and the leftmost column of π consists of a single box, $\Delta_{1,l-1} : W_\pi \to U(\mathfrak{gl}_1) \otimes W_{\pi_2}$ corresponds exactly to the map $Y^l_\sigma \to U(\mathfrak{gl}_1) \otimes Y^{l-1}_{\sigma_-}$ induced by $\Delta_- : Y_\sigma \to U(\mathfrak{gl}_1) \otimes Y_{\sigma_-}$.

Using (8.1), we can make sense of tensor products: if we are given W_{π_c}-modules V_c for each $c = 1, \dots, d$ then we obtain a well-defined W_π-module

$$V_1 \otimes \cdots \otimes V_d := \Delta^*_{l_1,\dots,l_d}(V_1 \boxtimes \cdots \boxtimes V_d), \tag{8.2}$$

i.e. we take the pull-back of their outer tensor product (viewed as a module via the usual sign convention).

Now specialize to the situation that $d = l$ and $l_1 = \cdots = l_d = 1$. Then each pyramid π_c is a single column of height one or two. In the former case $W_{\pi_c} = U(\mathfrak{gl}_1)$ and in the latter $W_{\pi_c} = U(\mathfrak{gl}_{1|1})$. So we have that $W_{\pi_1} \otimes \cdots \otimes W_{\pi_l} = U_\sigma^l$, and the map $\Delta_{1,\dots,1}$ coincides with the Miura transform μ.

Given $A \in \mathrm{Tab}_\pi$, let $A_c \in \mathrm{Tab}_{\pi_c}$ be its cth column and $\overline{L}(A_c)$ be the corresponding irreducible W_{π_c}-module. Let us decode this notation a little. If $W_{\pi_c} = U(\mathfrak{gl}_1)$ then A_c has just a single entry b and $\overline{L}(A_c)$ is the one-dimensional module with an even basis vector v_+ such that $e_{1,1}v_+ = (-1)^{|2|}bv_+$. If $W_{\pi_c} = U(\mathfrak{gl}_{1|1})$ then A_c has two entries, a in the first row and b in the second row, and $\overline{L}(A_c)$ is one- or two-dimensional according to whether $a = b$ or not; in both cases $\overline{L}(A_c)$ is generated by an even vector v_+ such that $e_{1,1}v_+ = (-1)^{|1|}av_+, e_{2,2}v_+ = (-1)^{|2|}bv_+$ and $e_{1,2}v_+ = 0$. Let

$$\overline{V}(A) := \overline{L}(A_1) \otimes \cdots \otimes \overline{L}(A_l). \tag{8.3}$$

Note that $\dim \overline{V}(A) = 2^{k-h}$ where h is the number of $c = 1, \dots, l$ such that A_c has two equal entries.

Lemma 8.3. *For any $A \in \mathrm{Tab}_\pi$, there is a non-zero homomorphism*

$$\overline{M}(A) \to \overline{V}(A)$$

sending the cyclic vector $1 \otimes 1_A \in \overline{M}(A)$ to $v_+ \otimes \cdots \otimes v_+ \in \overline{V}(A)$. In particular, $\overline{V}(A)$ contains a subquotient isomorphic to $\overline{L}(A)$.

Proof. Suppose that $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l}$. By the definition of $\overline{M}(A)$ as an induced module, it suffices to show that $v := v_+ \otimes \cdots \otimes v_+ \in \overline{V}(A)$ is annihilated by all $e^{(r)}$ for $r > s_{1,2}$ and that $d_1^{(r)}v = e_r(a_1, \dots, a_k)v$ and $d_2^{(r)}v = e_r(b_1, \dots, b_l)v$ for all $r > 0$. For this we calculate from the explicit formulae for the invariants $d_1^{(r)}, d_2^{(r)}$ and $e^{(r)}$ given by (4.10) and (4.19)–(4.21), remembering that their action on v is defined via the Miura transform $\mu = \Delta_{1,\dots,1}$. It is convenient in this proof to set

$$\bar{e}_{i,j}^{[c]} := \begin{cases} (-1)^{|i|}1^{\otimes(c-1)} \otimes e_{i,j} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 2, \\ (-1)^{|2|}1^{\otimes(c-1)} \otimes e_{1,1} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 1 \text{ and } i = j = 2, \\ 0 & \text{otherwise,} \end{cases}$$

for any $1 \le i, j \le 2$ and $1 \le c \le l$, where q_c is the number of boxes in the cth column of π. First we have that

$$d_1^{(r)}v = \sum_{1 \le c_1, \dots, c_r \le l} \sum_{1 \le h_1, \dots, h_{r-1} \le 2} \bar{e}_{1,h_1}^{[c_1]} \bar{e}_{h_1,h_2}^{[c_2]} \cdots \bar{e}_{h_{r-1},1}^{[c_r]} v$$

summing only over terms with $c_1 < \cdots < c_r$. The elements on the right commute (up to sign) because the c_i are all distinct, so any $\bar{e}_{1,2}^{[c_i]}$ produces zero as $e_{1,2}v_+ = 0$. Thus the summation reduces just to

$$\sum_{1 \le c_1 < \cdots < c_r \le l} \bar{e}_{1,1}^{[c_1]} \cdots \bar{e}_{1,1}^{[c_r]} v = e_r(a_1, \dots, a_k)v$$

as required. Next we have that

$$d_2^{(r)}v = \sum_{1\leq c_1,\dots,c_r\leq l}\ \sum_{1\leq h_1,\dots,h_{r-1}\leq 2} (-1)^{\#\{i=1,\dots,r-1\ |\ \mathrm{row}(h_i)=1\}} \bar{e}_{2,h_1}^{[c_1]}\bar{e}_{h_1,h_2}^{[c_2]}\cdots \bar{e}_{h_{r-1},2}^{[c_r]}v$$

summing only over terms with $c_i \geq c_{i+1}$ if $\mathrm{row}(h_i) = 1$, $c_i < c_{i+1}$ if $\mathrm{row}(h_i) = 2$. Here, if any monomial $\bar{e}_{1,2}^{[c_i]}$ appears, the rightmost such can be commuted to the end, when it acts as zero. Thus the summation reduces just to the terms with $h_1 = \cdots = h_{r-1} = 2$ and again we get the required elementary symmetric function $e_r(b_1,\dots,b_l)$. Finally we have that

$$e^{(r)}v = \sum_{1\leq c_1,\dots,c_r\leq l}\ \sum_{1\leq h_1,\dots,h_{r-1}\leq 2} (-1)^{\#\{i=1,\dots,r-1\ |\ \mathrm{row}(h_i)=1\}} \bar{e}_{1,h_1}^{[c_1]}\bar{e}_{h_1,h_2}^{[c_2]}\cdots \bar{e}_{h_{r-1},2}^{[c_r]}v$$

summing only over terms with $c_i \geq c_{i+1}$ if $\mathrm{row}(h_i) = 1$, $c_i < c_{i+1}$ if $\mathrm{row}(h_i) = 2$. As before this is zero because the rightmost $\bar{e}_{1,2}^{[c_i]}$ can be commuted to the end. □

Theorem 8.4. *Take any* $A = {}^{a_1\cdots a_k}_{b_1\cdots b_l} \in \mathrm{Tab}_\pi$ *and let* $h \geq 0$ *be maximal such that there exist distinct* $1 \leq i_1,\dots,i_h \leq k$ *and distinct* $1 \leq j_1,\dots,j_h \leq l$ *with* $a_{i_1} = b_{j_1},\dots,a_{i_h} = b_{j_h}$. *Choose* $B \sim A$ *so that* B *has* h *columns of height two containing equal entries. Then*

$$\overline{L}(A) \cong \overline{V}(B). \tag{8.4}$$

In particular, $\dim \overline{L}(A) = 2^{k-h}$.

Proof. By Lemma 8.3, $\overline{V}(B)$ has a subquotient isomorphic to $\overline{L}(B) \cong \overline{L}(A)$, which implies that $\dim \overline{L}(A) \leq \dim \overline{V}(B) = 2^{k-h}$. Also by the linear independence established in the partial proof of Theorem 7.3 given in the previous section we know that $\dim \overline{L}(A) \geq 2^{k-h}$. □

Theorem 8.4 also establishes the fact about dimension needed to complete the proof of Theorem 7.3 in the previous section.

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