Proceedings of the London Mathematical Society Advance Access published November 27, 2006

Proc. London Math. Soc. Page 1 of 26
doi:10.1112/plms/pdl009

GOOD GRADING POLYTOPES

JONATHAN BRUNDAN and SIMON M. GOODWIN

Abstract

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and $e \in \mathfrak{g}$ a nilpotent element. Elashvili and Kac have recently classified all good $\mathbb{Z}$-gradings for e. We instead consider *good* $\mathbb{R}$*-gradings*, which are naturally parameterized by an open convex polytope in a Euclidean space arising from the reductive part of the centralizer of e in $\mathfrak{g}$. As an application, we prove that the isomorphism type of the *finite* W*-algebra* attached to a good $\mathbb{R}$-grading for e is independent of the particular choice of good grading.

1. *Introduction*

In this article, we construct isomorphisms between the finite W-algebras associated to a nilpotent orbit in a complex semisimple Lie algebra. In some important special cases, these finite W-algebras were first defined and studied in the PhD Thesis of Lynch [**14**], generalizing a construction of Kostant [**13**]. The same algebras were later rediscovered by mathematical physicists, who coined the name 'finite W-algebra' used here; see for example [**2**]. In full generality, a finite W-algebra associated to an arbitrary nilpotent orbit was introduced only recently by Premet [**18**], who views the resulting algebra as an enveloping algebra for the Slodowy slice through the nilpotent orbit in question; see also [**7**].

To review a slight generalization of Premet's definition in more detail, let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and let $e \in \mathfrak{g}$ be nilpotent. An $\mathbb{R}$-grading

$$\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$$

of $\mathfrak{g}$ is called a *good grading* for e if $e \in \mathfrak{g}_2$ and the linear map $\operatorname{ad} e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for all $j \leqslant -1$ and surjective for all $j \geqslant -1$. This definition originates in [**12**]. We call a good grading *integral* if $\mathfrak{g}_j = 0$ for all $j \notin \mathbb{Z}$ and *even* if $\mathfrak{g}_j = 0$ for all $j \notin 2\mathbb{Z}$; these are the most important cases. A classification of all integral good gradings can be found in [**6**]. By [**6**, Theorem 2.1], even good gradings correspond to *nice parabolic subalgebras* as have been independently classified by Baur and Wallach [**1**].

Suppose Γ is a good grading for e, and let $(\,,\,)$ denote the Killing form on $\mathfrak{g}$. The alternating bilinear form $\langle\,,\,\rangle$ on $\mathfrak{g}_{-1}$ defined by $\langle x, y\rangle = ([x, y], e)$ is non-degenerate. Choose a Lagrangian subspace $\mathfrak{k}$ of $\mathfrak{g}_{-1}$ and define

$$\mathfrak{m} = \mathfrak{k} \oplus \bigoplus_{j < -1} \mathfrak{g}_j.$$

This is a nilpotent subalgebra of $\mathfrak{g}$ and the map $\chi : \mathfrak{m} \to \mathbb{C}$, $x \mapsto (x, e)$ defines a representation of $\mathfrak{m}$. The *finite* W*-algebra* associated to e and the good grading Γ may then be defined as the endomorphism algebra

$$H_\chi = \operatorname{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{\mathrm{op}}$$

Received 13 October 2005; revised 29 March 2006.

2000 *Mathematics Subject Classification* 17B20, 17B37.

The research of the first author was supported in part by NSF grant no. DMS-0139019, and that of the second author was supported in part by LIEGRITS and CAALT.

of the *generalized Gelfand–Graev representation* $U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$. A critical point for this article is that the definition of the algebra H_χ is independent of the particular choice of the Lagrangian subspace $\mathfrak{k}$. More precisely, given another Lagrangian subspace $\mathfrak{k}'$ of $\mathfrak{g}_{-1}$, a construction due to Gan and Ginzburg [**7**] (generalized slightly in Theorem 27 below) yields a canonical isomorphism between the finite W-algebras H_χ and $H_{\chi'}$ arising from the choices $\mathfrak{k}$ and $\mathfrak{k}'$, respectively.

As explained in detail in the introduction of [**3**], the algebras considered originally by Kostant and Lynch in [**13**, **14**] are naturally identified with the algebras H_χ defined here in the special case that the good grading Γ is even, that is, when the good grading arises from a nice parabolic subalgebra. In particular, in the even case, H_χ can actually be realized as a subalgebra of $U(\mathfrak{p})$, where $\mathfrak{p}$ is the parabolic subalgebra $\mathfrak{p} = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$. This makes the representation theory of H_χ easier to study in the even case; for instance, it is clear in these cases that H_χ possesses many finite-dimensional representations arising from restrictions of finite-dimensional $U(\mathfrak{p})$-modules. In general it is still an open problem to show even that H_χ has a 1-dimensional representation; see [**19**, Conjecture 3.1].

On the other hand, the algebras studied by Premet [**18**, **19**] and Gan and Ginzburg [**7**] are the algebras H_χ defined here in the special case that the good grading Γ is the *Dynkin grading*, that is, the grading defined by embedding e into an $\mathfrak{sl}_2$-triple (e, h, f) and considering the $\operatorname{ad} h$-eigenspace decomposition of $\mathfrak{g}$. Representation theory of $\mathfrak{sl}_2$ implies that the Dynkin grading is always an integral good grading for e. The present definition of H_χ, involving an arbitrary choice of good grading for e, gives a general framework containing both the Kostant–Lynch construction and the Premet construction as special cases. Our main result shows that in fact the algebra H_χ only depends up to isomorphism on e, not on the choice of good grading for e.

THEOREM 1. *The finite W-algebras H_χ and $H_{\chi'}$ associated to any two good gradings Γ and Γ' for e are isomorphic.*

To prove the theorem, we need to make precise the physicists' idea of deforming one good grading into another. Say two good gradings $\Gamma : \mathfrak{g} = \bigoplus_{i \in \mathbb{R}} \mathfrak{g}_i$ and $\Gamma' : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}'_j$ are *adjacent* if

$$\mathfrak{g} = \bigoplus_{i^- \leqslant j \leqslant i^+} \mathfrak{g}_i \cap \mathfrak{g}'_j,$$

where i^- denotes the largest integer strictly smaller than i and i^+ denotes the smallest integer strictly greater than i. If Γ and Γ' are adjacent, then by Lemma 26 below, there exist Lagrangian subspaces $\mathfrak{k}$ in $\mathfrak{g}_{-1}$ and $\mathfrak{k}'$ in $\mathfrak{g}'_{-1}$ such that

$$\mathfrak{k} \oplus \bigoplus_{i < -1} \mathfrak{g}_i = \mathfrak{k}' \oplus \bigoplus_{j < -1} \mathfrak{g}'_j,$$

that is, the nilpotent subalgebra $\mathfrak{m}$ defined from Γ and $\mathfrak{k}$ coincides with the nilpotent subalgebra $\mathfrak{m}'$ defined from Γ' and $\mathfrak{k}'$. With these choices, the algebras H_χ and $H_{\chi'}$ corresponding to Γ and Γ' are simply *equal*. In view of the aforementioned result of Gan and Ginzburg (independence of choice of Lagrangian subspace), Theorem 1 therefore follows if we can prove that any two good gradings for e are linked by a chain of adjacent good gradings. The precise statement is as follows.

THEOREM 2. *Given any two good gradings Γ and Γ' for e, there exists a chain $\Gamma_1, \dots, \Gamma_n$ of good gradings for e such that Γ is conjugate to Γ_1, Γ_i is adjacent to Γ_{i+1} for each $i = 1, \dots, n-1$, and Γ_n is conjugate to Γ'.*

To prove Theorem 2, there is a simple geometric picture. To explain, we need a little more notation. Pick an $\mathfrak{sl}_2$-triple (e, h, f) containing e and let $\mathfrak{s} = \langle e, h, f\rangle$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ containing h and $\Phi \subset \mathfrak{t}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The centralizers of e in $\mathfrak{g}$ and $\mathfrak{t}$ are denoted by $\mathfrak{g}_e$ and $\mathfrak{t}_e$, respectively.

Introduce the *restricted root system* $\Phi_e \subset \mathfrak{t}_e^*$, namely, the set of non-zero restrictions of roots $\alpha \in \Phi$ to $\mathfrak{t}_e$. We define the real form E_e of $\mathfrak{t}_e$ by

$$E_e = \{p \in \mathfrak{t}_e \mid \alpha(p) \in \mathbb{R} \text{ for all } \alpha \in \Phi_e\}.$$

Thus, E_e is a Euclidean space of dimension equal to the rank of the reductive part of $\mathfrak{g}_e$. For each $\alpha \in \Phi_e$, let $d(\alpha)$ denote the minimal dimension of an irreducible $\mathfrak{s}$-submodule of the α-weight space of $\mathfrak{g}$ with respect to $\mathfrak{t}_e$. The *good grading polytope* is then defined to be the open convex polytope

$$\mathscr{P}_e = \{p \in E_e \mid \alpha(p) < d(\alpha) \text{ for all } \alpha \in \Phi_e\};$$

see Example 22 below for an example.

For $p \in \mathscr{P}_e$, let $\Gamma(p)$ denote the $\mathbb{R}$-grading of $\mathfrak{g}$ defined by the eigenspace decomposition of the linear map $\mathrm{ad}(h + p) : \mathfrak{g} \to \mathfrak{g}$. By Theorem 20 below, this is a good $\mathbb{R}$-grading for e, and conversely every good $\mathbb{R}$-grading for e is conjugate to $\Gamma(p)$ for some $p \in \mathscr{P}_e$. The affine hyperplanes

$$H_{\alpha,k} = \{p \in E_e \mid \alpha(p) = k\}$$

for all $\alpha \in \Phi_e$ and $k \in \mathbb{Z}$ cut the good grading polytope into finitely many connected *alcoves*. In Theorem 25, we show that good gradings $\Gamma(p)$ and $\Gamma(p')$ for $p, p' \in \mathscr{P}_e$ are adjacent if and only if p and p' lie in the closure of the same alcove. Since one can get from any point in $\mathscr{P}_e$ to any other by crossing finitely many walls, Theorem 2 follows easily from this description.

We note by Theorem 21 that there is a natural finite group W_e of symmetries of $\mathscr{P}_e$ such that for $p, p' \in \mathscr{P}(e)$, the good gradings $\Gamma(p)$ and $\Gamma(p')$ are conjugate if and only if p and p' lie in the same W_e-orbit. Thus, W_e-orbits on $\mathscr{P}_e$ parameterize conjugacy classes of good $\mathbb{R}$-gradings for e. The group W_e of symmetries of $\mathscr{P}_e$ is actually a well-known group: we have

$$W_e \cong N_W(W_J)/W_J$$

where W_J is the parabolic subgroup of the Weyl group W corresponding to the minimal Levi subalgebra of $\mathfrak{g}$ containing e according to the Bala–Carter theory. We point out especially Lemma 15 below which gives another sense in which these groups are 'almost' reflection groups, different to that of Howlett [**9**].

We expect that the restricted root systems Φ_e investigated here will also play a role in the representation theory of the finite W-algebras H_χ themselves.

The remainder of the paper is organized as follows. In Section 2 we define and study the restricted root system associated to a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. In particular, we explain how conjugacy classes of bases for the restricted root system are in one-to-one correspondence with conjugacy classes of parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. Next, in Section 3, we recall some basic properties of the centralizer $\mathfrak{g}_e$ of e in $\mathfrak{g}$, and explain its relationship to the restricted root system arising from the minimal Levi subalgebra of $\mathfrak{g}$ containing e. In Section 4 we prove that the conjugacy classes of good gradings for e are parameterized by the W_e-orbits on the good grading polytope $\mathscr{P}_e$. Then in Section 5 we consider the partition of $\mathscr{P}_e$ into alcoves and prove that the good gradings parameterized by points p and p' are adjacent if and only if p and p' lie in the closure of the same alcove. This completes the proofs of Theorems 1 and 2. The final three sections give explicit descriptions of the good grading polytopes for classical Lie algebras, following the approach of [**6**] closely.

2. *Restricted root systems*

Let G be a semisimple algebraic group over $\mathbb{C}$, T be a maximal torus, and B be a Borel subgroup containing T. We write $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{b}$ for the corresponding Lie algebras. Recall some standard notation:

- $\Phi \subset \mathfrak{t}^*$ denotes the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$;
- $\mathfrak{g}_\alpha$ denotes the α-root space of $\mathfrak{g}$ for each $\alpha \in \Phi$;
- $\Phi^+ \subseteq \Phi$ is the system of positive roots defined from $\mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$;
- $\Delta = \{\alpha_1, \dots, \alpha_r\}$ is the corresponding set of simple roots;
- E is the $\mathbb{R}$-lattice $\mathbb{R}\alpha_1 \oplus \dots \oplus \mathbb{R}\alpha_r$ in $\mathfrak{t}^*$ and E^* is the dual lattice in $\mathfrak{t}$;
- $H_\alpha = \ker \alpha$ is the hyperplane in E^* defined by $\alpha \in \Phi$;
- $\mathscr{A}$ is the hyperplane arrangement $\{H_\alpha \mid \alpha \in \Phi\}$ in the real vector space E^*;
- $W < \mathrm{GL}(E^*)$ is the Weyl group generated by the simple reflections $s_1, \dots, s_r$, where s_i is the reflection in the hyperplane $H_i = H_{\alpha_i}$.

Given, in addition, a subset J of $\{1, \dots, r\}$, we adopt some more standard notation for parabolic objects associated to J:

- E_J denotes $\sum_{j \in J} \mathbb{R}\alpha_j \subseteq E$;
- $\Phi_J = \Phi \cap E_J$ is the closed subsystem of Φ generated by $\{\pm\alpha_j \mid j \in J\}$ with base $\Delta_J = \{\alpha_j \mid j \in J\}$;
- $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$ denotes the standard parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}_J = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_J} \mathfrak{g}_\alpha$ and nilradical $\mathfrak{u}_J = \sum_{\alpha \in \Phi^+ \setminus \Phi_J} \mathfrak{g}_\alpha$;
- $P_J = L_J U_J$ is the corresponding standard parabolic subgroup of G with standard Levi subgroup L_J and unipotent radical U_J;
- W_J denotes the parabolic subgroup of W generated by $\{s_j \mid j \in J\}$.

Our final piece of notation is less standard: $E^J \cong (E/E_J)^*$ denotes $\bigcap_{j \in J} H_j \subseteq E^*$. Then,

$$\mathscr{A}^J = \{H_\alpha \cap E^J \mid \alpha \in \Phi \setminus \Phi_J\}$$

is the restriction of the reflection arrangement $\mathscr{A}$ to the subspace E^J. It has been well studied in the literature, starting from work of Orlik and Solomon [**15**]. For $\alpha \in E$, we let $\alpha^J \in E/E_J$ denote the restriction of α to E^J. In this section, we want to focus not on the restricted arrangement $\mathscr{A}^J$, but rather on the *restricted root system*

$$\Phi^J = \{\alpha^J \mid \alpha \in \Phi \setminus \Phi_J\}$$

consisting of all the non-zero restrictions of roots in Φ to E^J. The hyperplanes in $\mathscr{A}^J$ are the kernels of the restricted roots in Φ^J, so one can recover $\mathscr{A}^J$ from Φ^J, but not vice versa. Note that Φ^J is in general definitely *not* a root system in E/E_J in the usual sense.

From now on, we will always identify E with E^* using the real inner product $(\,,)$ induced by the Killing form on $\mathfrak{g}$. We can then identify both the spaces E^J and E/E_J with the orthogonal complement to E_J in E. Under this identification, the notation α^J becomes the orthogonal projection of $\alpha \in E$ to E^J along the direct sum decomposition $E = E^J \oplus E_J$. Let us also set $I = \{1, \dots, r\} \setminus J$ and $m = |I| = \dim E^J$.

LEMMA 3. *For any $\alpha \in \Phi^J$, there exists $\alpha' \in E_J$ such that $\alpha + \alpha' \in \Phi$ and $(\alpha', \alpha_j) \geqslant 0$ for all $j \in J$.*

Proof. By the definition of Φ^J, the set $\{\alpha' \in E_J \mid \alpha + \alpha' \in \Phi\}$ is non-empty. Pick an element α' from this set that is maximal in the dominance ordering. To complete the proof, we just need to show that $(\alpha', \alpha_j) \geqslant 0$ for all $j \in J$. Otherwise, we can find $j \in J$ such that $(\alpha + \alpha', \alpha_j) = (\alpha', \alpha_j) < 0$, but then $\alpha + (\alpha' + \alpha_j) \in \Phi$ by [**10**, Lemma 9.4] contradicting the maximality of the choice of α'. $\square$

LEMMA 4. *If $\alpha, \beta \in \Phi^J$ are distinct roots with $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi^J$ too.*

Proof. By the previous lemma, we can lift α and β to $\alpha + \alpha', \beta + \beta' \in \Phi$, where $\alpha', \beta' \in E_J$ satisfy $(\alpha', \alpha_j) \geqslant 0$ and $(\beta', \alpha_j) \geqslant 0$ for all $j \in J$. In other words, α' and β' belong to the closure of the same chamber in E_J; hence $(\alpha', \beta') \geqslant 0$. So

$$(\alpha + \alpha', \beta + \beta') = (\alpha, \beta) + (\alpha', \beta') > 0.$$

Also $\alpha + \alpha' \neq \beta + \beta'$ since $\alpha \neq \beta$. So [**10**, Lemma 9.4] implies that $(\alpha + \alpha') - (\beta + \beta') \in \Phi$. Hence, $\alpha - \beta \in \Phi^J$. □

LEMMA 5. *If $\alpha, \beta \in \Phi^J$ are proportional roots, then there exists $\gamma \in \Phi^J$ such that α and β are both integer multiples of γ.*

Proof. Let $M = \{c > 0 \mid c\alpha \in \Phi^J\}$. The previous lemma implies that if a and b are distinct elements of M then $|a - b| \in M$ too. It follows that any element of M is an integer multiple of the smallest element. □

Define a *base* of the restricted root system Φ^J to be a subset $\{\beta_i \mid i \in I\}$ of Φ^J such that any element of Φ^J can be written as $\sum_{i \in I} a_i \beta_i$ with either all $a_i \in \mathbb{Z}_{\geqslant 0}$ or all $a_i \in \mathbb{Z}_{\leqslant 0}$. Of course any base for Φ^J is necessarily a basis for the vector space E^J. Any base $\{\beta_i \mid i \in I\}$ partitions the restricted root system Φ^J into positive and negative roots, the positive ones being the roots that are a positive linear combination of several β_i. In order to construct bases of Φ^J in the usual way, let $\gamma \in E^J$ be *regular*. This means that γ does not lie on any of the hyperplanes in $\mathscr{A}^J$, or equivalently, $(\alpha, \gamma) \neq 0$ for all $\alpha \in \Phi^J$. Then we can define $\Phi^J(\gamma)$ to be $\{\alpha \in \Phi^J \mid (\alpha, \gamma) > 0\}$, and clearly $\Phi^J = \Phi^J(\gamma) \sqcup (-\Phi^J(\gamma))$. Call $\alpha \in \Phi^J(\gamma)$ *decomposable* if α can be written as $\alpha = \beta_1 + \beta_2$ for $\beta_1, \beta_2 \in \Phi^J(\gamma)$, and *indecomposable* if it is not decomposable. Armed with Lemmas 4 and 5, one can prove the following theorem in essentially the same way as for root systems; see for example [**10**, Theorem 10.1].

THEOREM 6. *Let $\gamma \in E^J$ be regular. Then the set $\Delta^J(\gamma)$ of all indecomposable roots in $\Phi^J(\gamma)$ is a base of Φ^J, and every base can be obtained in this manner.*

Recall that the bases for the root system Φ are in natural bijective correspondence with the set $\mathscr{C}$ of *chambers* in the hyperplane arrangement $\mathscr{A}$, that is, the connected components of $E \setminus \bigcup \mathscr{A}$. Under this correspondence, the base $\{\beta_1, \ldots, \beta_r\}$ corresponds to the chamber

$$\{\alpha \in E \mid (\alpha, \beta_i) > 0 \text{ for all } i = 1, \ldots, r\}.$$

Theorem 6 leads to a similar bijection between the set of bases of the restricted root system Φ^J and the set $\mathscr{C}^J$ of chambers in the hyperplane arrangement $\mathscr{A}^J$.

COROLLARY 7. *There is a natural bijective correspondence between bases in Φ^J and chambers in $\mathscr{C}^J$, under which the base $\{\beta_i \mid i \in I\}$ corresponds to the chamber*

$$\{\alpha \in E^J \mid (\alpha, \beta_i) > 0 \text{ for all } i \in I\}.$$

Proof. We just explain how to construct the inverse map from chambers to bases. Given a chamber $C \in \mathscr{C}^J$, pick any (necessarily regular) point $\gamma \in C$. Then, the image of C under the inverse map is the base $\Delta^J(\gamma)$ for Φ^J. This is well defined, because if γ and γ' belong to the same chamber, then they lie on the same side of each hyperplane in $\mathscr{A}^J$, so $\Phi^J(\gamma) = \Phi^J(\gamma')$. □

There is another way to construct bases for the restricted root system Φ^J, by restricting bases for Φ that contain bases for Φ_J.

LEMMA 8. *Suppose that $\{\beta_1, \dots, \beta_r\}$ is a base for Φ such that $\{\beta_j \mid j \in J\}$ is a base for Φ_J. Then, $\{\beta_i^J \mid i \in I\}$ is a base for Φ^J, and every base for Φ^J can be obtained in this way.*

Proof. Suppose first that $\{\beta_1, \dots, \beta_r\}$ is a base for Φ such that $\{\beta_j \mid j \in J\}$ is a base for Φ_J. Any $\alpha \in \Phi \setminus \Phi_J$ can be written as $\alpha = \sum_{j=1}^r a_j \beta_j$, so that the a_j are either all greater than or equal to 0 or all less than or equal to 0. Since $\beta_j \in E_J$ for each $j \in J$, $\alpha^J = \sum_{i \in I} a_i \beta_i^J$. Hence, $\{\beta_i^J \mid i \in I\}$ is a base for Φ^J.

To show that every base in Φ^J arises in this way, we think instead in terms of chambers. Let $C \in \mathscr{C}$ be the chamber corresponding to the base $\{\beta_1, \dots, \beta_r\}$, still assuming that $\{\beta_j \mid j \in J\}$ is a base for Φ_J. The closure $\overline{C}$ is equal to

$$\{\alpha \in E \mid (\alpha, \beta_i) \geqslant 0 \text{ for all } i = 1, \dots, r\},$$

while $E^J = \{\alpha \in E \mid (\alpha, \beta_j) = 0 \text{ for all } j \in J\}$. Hence, the intersection $\overline{C} \cap E^J$ is equal to $\{\alpha \in E^J \mid (\alpha, \beta_i^J) \geqslant 0 \text{ for all } j \in J\}$. This shows that $(\overline{C} \cap E^J) \setminus \bigcup \mathscr{A}^J$ is the chamber in $\mathscr{C}^J$ corresponding to the base $\{\beta_i^J \mid i \in I\}$. We must prove that every chamber in $\mathscr{C}^J$ can be obtained in this way.

Suppose that $\{\beta_1, \dots, \beta_r\}$ is a base for Φ that does *not* contain a base for Φ_J, and let C be the corresponding chamber in $\mathscr{C}$. We can find $\beta = \sum_{j=1}^r a_j \beta_j \in \Phi_J$ such that $a_i \neq 0$ for some $1 \leqslant i \leqslant r$ with $\beta_i \notin \Phi_J$. Take any $\alpha \in \overline{C} \cap E^J$, so $(\alpha, \beta) = 0$ and $(\alpha, \beta_j) \geqslant 0$ for all $j = 1, \dots, r$. Since $a_i \neq 0$, the equation $\sum_{j=1}^r a_j(\alpha, \beta_j) = 0$ implies that $(\alpha, \beta_i) = 0$. Hence, $\overline{C} \cap E^J$ is contained in the hyperplane H_{β_i}, and $(\overline{C} \cap E^J) \setminus \bigcup \mathscr{A}^J = \emptyset$. Since $E^J \setminus \bigcup \mathscr{A}^J$ is obviously covered by the sets $(\overline{C} \cap E^J) \setminus \bigcup \mathscr{A}^J$ as C runs over all chambers in $\mathscr{A}$, we have now shown that every chamber in $\mathscr{C}^J$ is equal to $(\overline{C} \cap E^J) \setminus \bigcup \mathscr{A}^J$ for some chamber C in $\mathscr{C}$ such that the corresponding base of Φ contains a base for Φ_J. □

LEMMA 9. *Suppose that $\{\beta_1, \dots, \beta_r\}$ and $\{\gamma_1, \dots, \gamma_r\}$ are two bases for Φ such that $\{\beta_j \mid j \in J\}$ and $\{\gamma_j \mid j \in J\}$ are bases for Φ_J. The resulting bases $\{\beta_i^J \mid i \in I\}$ and $\{\gamma_i^J \mid i \in I\}$ for Φ^J are equal if and only if there exists $w \in W_J$ mapping $\{\beta_1, \dots, \beta_r\}$ to $\{\gamma_1, \dots, \gamma_r\}$.*

Proof. Since W_J acts trivially on E^J, it is easy to see that if $\{\beta_1, \dots, \beta_r\}$ and $\{\gamma_1, \dots, \gamma_r\}$ are conjugate under W_J, then $\{\beta_i^J \mid i \in I\}$ and $\{\gamma_i^J \mid i \in I\}$ are equal. Conversely, suppose that $\{\beta_i^J \mid i \in I\}$ and $\{\gamma_i^J \mid i \in I\}$ are equal. Recalling that W_J acts transitively on bases for Φ_J, we can conjugate and reindex if necessary to assume that $\beta_j = \gamma_j$ for all $j \in J$ and that $\beta_i^J = \gamma_i^J$ for all $i \in I$. But then we can certainly write

$$\beta_i = \gamma_i + \sum_{j \in J} a_{i,j} \gamma_j$$

for every $i \in I$ and scalars $a_{i,j} \in \mathbb{R}$. Since β_i is a root and the γ_i form a base for Φ, we get $a_{i,j} \geqslant 0$ for all $i \in I$ and $j \in J$. However, also

$$\gamma_i = \beta_i - \sum_{j \in J} a_{i,j} \beta_j$$

for every $i \in I$, which implies that all $a_{i,j} \leqslant 0$ too. Hence, $\beta_i = \gamma_i$ for each $i \in I$, and the original bases for Φ are equal as required. □

THEOREM 10. *There is a natural bijective correspondence between bases for Φ containing Δ_J and bases for Φ^J, under which the base $\{\beta_i, \alpha_j \mid i \in I, j \in J\}$ for Φ corresponds to the base $\{\beta_i^J \mid i \in I\}$ for Φ^J.*

Proof. Since W_J acts simply transitively on bases for Φ_J, each W_J-orbit of bases $\{\beta_1, \ldots, \beta_r\}$ for Φ containing a base for Φ_J has a unique representative that contains Δ_J. Given this, the theorem is immediate from Lemmas 8 and 9. $\square$

We remark that bases for Φ containing Δ_J are also in bijective correspondence with parabolic subgroups P of G that have L_J as a Levi factor, the base $\{\beta_i, \alpha_j \mid i \in I, j \in J\}$ for Φ corresponding to the parabolic subgroup with Lie algebra generated by $\mathfrak{l}_J$ and all $\mathfrak{g}_{\beta_i}$ $(i \in I)$. So another way of thinking about Theorem 10 is that choosing a base for the restricted root system Φ^J is equivalent to choosing a parabolic subgroup P of G with Levi factor L_J, just as choosing a base for Φ is equivalent to choosing a Borel subgroup of G containing T.

Corresponding to the base Δ of Φ, or to the standard parabolic subgroup P_J of G, we have the *standard base*

$$\Delta^J = \{\alpha^J \mid \alpha \in \Delta \setminus \Delta_J\} = \{\alpha_i^J \mid i \in I\}$$

of Φ^J. Now suppose that K is a subset of $\{1, \ldots, r\}$ such that $w \cdot \Delta_K = \Delta_J$ for some $w \in W$. Since $w \cdot E_K = E_J$, w induces an isometry between E^K and E^J which maps Φ^K to Φ^J. So if we apply w to the standard base Δ^K of Φ^K, we obtain a base $w \cdot \Delta^K$ for Φ^J. Clearly, all bases for Φ containing Δ_J are of the form $w \cdot \Delta$ for some $K \subset \{1, \ldots, r\}$ and some $w \in W$ such that $w \cdot \Delta_K = \Delta_J$. Therefore, by Theorem 10, all bases for Φ^J are of the form $w \cdot \Delta^K$ for suitable w and K. This means that for most purposes, it is sufficient to work only with standard bases Δ^J, providing one is prepared to allow the subset J of $\{1, \ldots, r\}$ to change.

Finally, we introduce the *restricted Weyl group* W^J, namely, the stabilizer in W of the set Δ_J. This is a well-known group, studied in particular by Howlett [**9**]; see also [**4**, § 10.4]. Clearly, W^J normalizes W_J and $W^J \cap W_J = \{1\}$. In fact, by [**9**, Lemma 2], we have $W_J W^J = N_W(W_J)$, so $W^J \cong N_W(W_J)/W_J$. By Lemma 11 below, the natural action of W^J on E^J is faithful, so we can view W^J as a subgroup of $\mathrm{GL}(E^J)$. In general, W^J is *not* a reflection group, though it is close to being one in a sense made precise in Howlett's work; we will give an alternative explanation of this phenomenon in the next section. Clearly, W^J leaves the subset $\Phi^J \subset E^J$ invariant; hence we get an induced action of W^J on the set of bases for the root system Φ^J.

For the next lemma we require the following piece of notation: define $\mathscr{K}_J$ to be the set of subsets K of $\{1, \ldots, r\}$ with the property that $w \cdot \Delta_K = \Delta_J$ for some $w \in W$.

LEMMA 11. *For each $K \in \mathscr{K}_J$, pick $w_K \in W$ such that $w_K \cdot \Delta_K = \Delta_J$. Then,*

$$\{w_K \cdot \Delta^K \mid K \in \mathscr{K}_J\}$$

is a set of orbit representatives for the action of the restricted Weyl group W^J on the set of bases for Φ^J. Moreover, each orbit is regular, of size $|W^J|$.

Proof. The set of all $w \in W$ with the property that $\Delta_J \subseteq w \cdot \Delta$ is the disjoint union $\bigcup_{K \in \mathscr{K}_J} W^J w_K$. Since W acts simply transitively on bases for Φ, this means that there are $|\mathscr{K}_J||W^J|$ different bases for Φ containing Δ_J, namely, the bases $\{w w_K \cdot \Delta \mid w \in W^J, K \in \mathscr{K}_J\}$. Applying Theorem 10, we deduce that there are $|\mathscr{K}_J||W^J|$ different bases for Φ^J, namely, the bases $\{w w_K \cdot \Delta^K \mid w \in W^J, K \in \mathscr{K}_J\}$. The lemma follows. $\square$

This lemma immediately implies that the number of bases for the restricted root system Φ^J is equal to $|\mathscr{K}_J||W^J|$. Equivalently, by Corollary 7, the number of chambers in the hyperplane arrangement $\mathscr{A}^J$ is given by the formula

$$\left|\mathscr{C}^J\right| = |\mathscr{K}_J| \left|W^J\right|.$$

This is a well-known identity due originally to Orlik and Solomon [**15**, (4.2)]. The hyperplane arrangement $\mathscr{A}^J$ is known to be a free arrangement; see [**17**, **5**]. So by [**16**, § 4.6] its Poincaré polynomial can be expressed as $(1 + b_1^J t) \dots (1 + b_m^J t)$ for *exponents* $b_1^J \leqslant \dots \leqslant b_m^J$. This factorization already appears in [**15**], and the exponents were computed there too in all cases. It is well known that

$$\left|\mathscr{A}^J\right| = b_1^J + b_2^J + \dots + b_m^J \quad \text{and} \quad \left|\mathscr{C}^J\right| = (1 + b_1^J)(1 + b_2^J) \dots (1 + b_m^J).$$

Moreover, if G is simple and $m \geqslant 1$ then the arrangement $\mathscr{A}^J$ is irreducible; hence $b_1^J = 1$ and, assuming $m \geqslant 2$ too, $b_2^J \geqslant 2$.

To conclude the section, we want to mention a theorem of Sommers [**20**] which gives a quick way to determine the exponents b_i^J. For any $\alpha = \sum_{i=1}^r a_i \alpha_i \in E$, we let $\mathrm{ht}(\alpha)$ denote $\sum_{i=1}^r a_i$. Let $\theta = \sum_{i=1}^r c_i \alpha_i$ be the highest root in Φ, and recall that all other roots in Φ are strictly smaller than θ in the dominance ordering. It follows easily that θ^J is the unique highest root in Φ^J. Now introduce the following plausible analogue of the Coxeter number for the restricted root system Φ^J: let

$$h^J = \min\{\mathrm{ht}(\theta^K) + 1 \mid K \in \mathscr{K}_J\}.$$

Then, Sommers' theorem says that an integer $1 \leqslant p < h^J$ belongs to the set $\{b_1^J, \dots, b_m^J\}$ of exponents whenever it is prime to all the coefficients $c_1, \dots, c_r$ of θ. Combined with the facts mentioned in the previous paragraph, and also [**9**] (or Lemmas 14–15 below) from which the orders of the groups W^J can be computed, this always gives enough information to allow one to completely determine the exponents.

Example 12. Take G to be a simple group of type E_7; we write $G = E_7$ for short. Label the simple roots, which we identify with the vertices of the Dynkin diagram, as follows:

$$\begin{array}{cccccc} 3 & 4 & 2 & 5 & 6 & 7 \\ & & 1 & & & \end{array}$$

Take $I = \{1, 2\}$ and $J = \{3, 4, 5, 6, 7\}$, so L_J is of type $A_3 + A_2$. The positive roots in Φ^J corresponding to the standard base $\Delta^J = \{\alpha_1^J, \alpha_2^J\}$ are

$$\{\alpha_1^J, \alpha_2^J, \alpha_1^J + \alpha_2^J, \alpha_1^J + 2\alpha_2^J, \alpha_1^J + 3\alpha_2^J, 2\alpha_1^J + 3\alpha_2^J, 2\alpha_1^J + 4\alpha_2^J\}.$$

The restricted Cartan matrix with ij-entry

$$\frac{2(\alpha_i^J, \alpha_j^J)}{(\alpha_j^J, \alpha_j^J)} \quad \text{for } i, j = 1, 2$$

is the matrix

$$\begin{pmatrix} 2 & -\frac{24}{7} \\ -1 & 2 \end{pmatrix}.$$

It follows that $\alpha_1^J \perp (\alpha_1^J + 2\alpha_2^J)$. We get the picture of roots and orthogonal hyperplanes shown in Figure 1.

There are twelve chambers in the hyperplane arrangement $\mathscr{A}^J$, the one corresponding to the standard base being shaded. Since $|\mathscr{K}_J| = 3$, there are three W^J-orbits on chambers. In fact, $W^J \cong S_2 \times S_2$ is generated by the reflections in the horizontal and vertical axes. The highest root θ^J is $2\alpha_1^J + 4\alpha_2^J$, but the Coxeter number h^J is 6 not 7: it comes from the highest root $2\alpha_1^J + 3\alpha_2^J$ with respect to the non-standard base $\{\alpha_1^J + 3\alpha_2^J, -\alpha_2^J\}$. The exponents are 1 and 5.

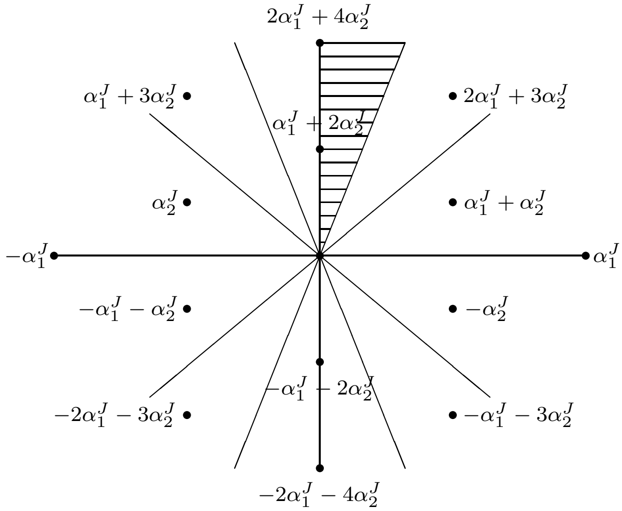


FIGURE 1

3. *Centralizers*

We fix for the remainder of the article a nilpotent element $e \in \mathfrak{g}$; our basic references for all matters concerning nilpotent orbits are [**11**, Chapters 1–5] and [**4**, Chapter 5]. We denote the centralizer of e in G either by $Z_G(e)$ or by G_e for short. Similarly, we write $\mathfrak{z}_{\mathfrak{g}}(e)$ or $\mathfrak{g}_e$ for its centralizer in $\mathfrak{g}$. By the Jacobson–Morozov theorem, we can embed e into an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s} = \langle e, h, f \rangle$, so that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Moreover, by a result of Kostant, any other such triple (e, h', f') is conjugate to (e, h, f) by an element of the connected centralizer G_e°.

The $\operatorname{ad} h$-eigenspace decomposition of $\mathfrak{g}$ defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ which we call the *Dynkin grading*. Let $\mathfrak{c} = \mathfrak{g}_0$ and let C be the corresponding closed connected subgroup of G. In other words, $\mathfrak{c}$ and C are the centralizers of h in $\mathfrak{g}$ and G, respectively. Also let $\mathfrak{r} = \bigoplus_{j>0} \mathfrak{g}_j$ and let R be the corresponding closed connected subgroup of G. It is well known that C_e is a maximal reductive subgroup of G_e, with Lie algebra $\mathfrak{c}_e$, and that R_e is the unipotent radical of G_e, with Lie algebra $\mathfrak{r}_e$. Moreover, G_e is the semidirect product $C_e \ltimes R_e$, and $\mathfrak{g}_e$ is the semidirect sum $\mathfrak{c}_e \oplus \mathfrak{r}_e$. Finally, the component group G_e/G_e° is isomorphic to C_e/C_e°.

Fix a maximal torus T of G contained in C and containing a maximal torus of C_e. An important role is played by the centralizer $\mathfrak{t}_e$ of e in the Lie algebra $\mathfrak{t}$ of T. It is a Cartan subalgebra of the reductive part $\mathfrak{c}_e$ of the centralizer $\mathfrak{g}_e$. Let L be the centralizer of $\mathfrak{t}_e$ in G, and let $\mathfrak{l}$ be the Lie algebra of L, that is, the centralizer of $\mathfrak{t}_e$ in $\mathfrak{g}$. Thus, L is a Levi subgroup of G, and the centre of $\mathfrak{l}$ is equal to $\mathfrak{t}_e$. By the Bala–Carter theory, $\mathfrak{l}$ is a minimal Levi subalgebra of $\mathfrak{g}$ containing e, and e is a distinguished nilpotent element of the derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$. Moreover, both h and f automatically lie in $[\mathfrak{l}, \mathfrak{l}]$.

LEMMA 13. *The set of weights of $\mathfrak{t}_e$ on $\mathfrak{g}_e$ equals the set of weights of $\mathfrak{t}_e$ on $\mathfrak{g}$.*

Proof. For $\alpha \in \mathfrak{t}_e^*$ and $i \geqslant 0$, let $L(\alpha, i)$ denote the irreducible $\mathfrak{t}_e \oplus \mathfrak{s}$-module of dimension $(i+1)$ on which $\mathfrak{t}_e$ acts by weight α. Decompose $\mathfrak{g}$ as a $\mathfrak{t}_e \oplus \mathfrak{s}$-module

$$\mathfrak{g} \cong \bigoplus_{\alpha \in \mathfrak{t}_e^*} \bigoplus_{i \geqslant 0} m(\alpha, i) L(\alpha, i)$$

for multiplicities $m(\alpha, i) \geqslant 0$. The set of weights of $\mathfrak{t}_e$ on $\mathfrak{g}$ is

$$\{\alpha \in \mathfrak{t}_e^* \mid m(\alpha, i) \neq 0 \text{ for some } i \geqslant 0\}.$$

Since each $L(\alpha, i)$ contains a non-zero vector annihilated by e, this is also the set of weights of $\mathfrak{t}_e$ on $\mathfrak{g}_e$. □

We define $\Phi_e \subset \mathfrak{t}_e^*$ to be the set of all non-zero weights of $\mathfrak{t}_e$ on $\mathfrak{g}_e$. The zero-weight space of $\mathfrak{t}_e$ on $\mathfrak{g}_e$ is of course the centralizer $\mathfrak{l}_e$ of e in the Levi subalgebra $\mathfrak{l}$. So we have the following analogue of the Cartan decomposition for centralizers:

$$\mathfrak{g}_e = \mathfrak{l}_e \oplus \bigoplus_{\substack{\alpha \in \Phi_e \\ i \geqslant 0}} \mathfrak{g}_e(\alpha, i)$$

where $\mathfrak{g}_e(\alpha, i) = \{x \in \mathfrak{g}_e \mid [h, x] = ix \text{ and } [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{t}_e\}$. This decomposition is compatible with the decomposition $\mathfrak{g}_e = \mathfrak{c}_e \oplus \mathfrak{r}_e$; indeed, we have

$$\mathfrak{c}_e = \mathfrak{t}_e \oplus \bigoplus_{\alpha \in \Phi_e^\circ} \mathfrak{g}_e(\alpha, 0), \qquad \mathfrak{r}_e = [\mathfrak{l}, \mathfrak{l}]_e \oplus \bigoplus_{\substack{\alpha \in \Phi_e \\ i > 0}} \mathfrak{g}_e(\alpha, i),$$

where Φ_e° denotes the set of all $\alpha \in \Phi_e$ such that $\mathfrak{g}_e(\alpha, 0)$ is non-zero. The root system Φ_e is a restricted root system in the sense of the previous section. To explain this, we need to make one more important choice: let P be a parabolic subgroup of G such that L is a Levi factor of P. Let U be the unipotent radical of P, and denote the corresponding Lie algebras by $\mathfrak{p}$ and $\mathfrak{u}$, so $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. Pick a Borel subgroup B of G contained in P and containing T. The choices of T and B determine a root system Φ for $\mathfrak{g}$ and a base $\Delta = \{\alpha_1, \dots, \alpha_r\}$, and we can appeal to the set-up from the previous section. We then have $L = L_J$, $U = U_J$ and $P = P_J$ for a unique subset J of $\{1, \dots, r\}$. The Euclidean space E^J from the previous section, henceforth denoted E_e, is an $\mathbb{R}$-form for the centre $\mathfrak{t}_e$ of the Lie algebra $\mathfrak{l} = \mathfrak{l}_J$, and Lemma 13 shows that Φ_e coincides with the restricted root system $\Phi^J \subseteq \mathfrak{t}_e^*$. The standard base Δ^J for Φ^J will be denoted from now on by Δ_e, and we let Φ_e^+ denote the corresponding set of positive roots. In the above root space decomposition of $\mathfrak{g}_e$, we have $\mathfrak{p}_e = \mathfrak{l}_e \oplus \mathfrak{u}_e$ where

$$\mathfrak{u}_e = \bigoplus_{\substack{\alpha \in \Phi_e^+ \\ i \geqslant 0}} \mathfrak{g}_e(\alpha, i).$$

Indeed, as explained in the previous section, the choice of the parabolic $P = LU$ is actually equivalent to this choice Φ_e^+ of positive roots in Φ_e.

The objects $\Phi_e, \Delta_e, \Phi_e^+, \dots$ introduced so far really only depend on J, with the exception of Φ_e° which does involve e itself: it is the root system of the reductive Lie algebra $\mathfrak{c}_e$. Let Δ_e° denote the base of Φ_e° associated to the positive system $\Phi_e^\circ \cap \Phi_e^+$. The *dominant chamber* of the hyperplane arrangement $\mathscr{A}_e^\circ$ in E_e associated to the root system Φ_e° is

$$\{\alpha \in E_e \mid (\alpha, \beta) > 0 \text{ for all } \beta \in \Delta_e^\circ\}.$$

It is usually *not* a single chamber in the hyperplane arrangement $\mathscr{A}_e$ defined by Φ_e, though it certainly contains the *standard chamber* $\{\alpha \in E_e \mid (\alpha, \beta) > 0 \text{ for all } \beta \in \Delta_e\}$ of $\mathscr{A}_e$.

This set-up gives a natural way to understand the restricted Weyl group $W^J \cong N_W(W_J)/W_J$ from the previous section. Recall that this acts faithfully on the vector space $E_e = E^J$, so extending scalars we can view W^J as a subgroup of $\mathrm{GL}(\mathfrak{t}_e)$. Let

$$W_e = N_{G_e}(\mathfrak{t}_e)/Z_{G_e}(\mathfrak{t}_e),$$

also naturally a subgroup of $\mathrm{GL}(\mathfrak{t}_e)$. Using the decomposition $G_e = C_e \ltimes R_e$ and noting that $\mathfrak{t}_e \subseteq \mathfrak{c}_e$, one can easily see that

$$N_{G_e}(\mathfrak{t}_e) = N_{C_e}(\mathfrak{t}_e) \ltimes Z_{R_e}(\mathfrak{t}_e) \quad \text{and} \quad Z_{G_e}(\mathfrak{t}_e) = Z_{C_e}(\mathfrak{t}_e) \ltimes Z_{R_e}(\mathfrak{t}_e).$$

Hence, we can also write $W_e = N_{C_e}(\mathfrak{t}_e)/Z_{C_e}(\mathfrak{t}_e)$ as subgroups of $\mathrm{GL}(\mathfrak{t}_e)$.

LEMMA 14. *As subgroups of* $\mathrm{GL}(\mathfrak{t}_e)$, *we have* $W_e = W^J$.

Proof. We identify W with $N_G(T)/T = N_G(\mathfrak{t})/T$. Take $x \in W^J$ represented by $\dot{x} \in N_G(T)$. Since $x \cdot \Delta_J = \Delta_J$, $\dot{x}$ normalizes L. Hence, $\dot{x} \cdot e$ is another distinguished nilpotent element of $[\mathfrak{l}, \mathfrak{l}]$. We claim that there exists $y \in L$ such that $\dot{x} \cdot e = y \cdot e$. To see this, it suffices by the classification of distinguished nilpotent orbits in $[\mathfrak{l}, \mathfrak{l}]$ to see that $\dot{x} \cdot e$ has the same labelled Dynkin diagram as e. This is true because by inspection of the tables in [**4**], the labelled Dynkin diagrams parameterizing distinguished nilpotent orbits of Levi subalgebras of simple Lie algebras are invariant under graph automorphisms. Hence, we have found an element $y^{-1}\dot{x} \in G_e$ which normalizes $\mathfrak{t}_e$ and acts on $\mathfrak{t}_e$ in the same way as x. This shows that $W^J \subseteq W_e$.

Conversely, take $x \in W_e$ represented by $\dot{x} \in N_{G_e}(\mathfrak{t}_e)$. Recalling that $L = Z_G(\mathfrak{t}_e)$, we note that $\dot{x}$ certainly normalizes L too. Now $\dot{x} \cdot T$ is a maximal torus of L, so there exists $y \in L$ such that $y^{-1}\dot{x} \in N_G(T)$. Since $\dot{x}$ normalizes $\mathfrak{l}$, it normalizes the centre $\mathfrak{t}_e$ of $\mathfrak{l}$, while y centralizes $\mathfrak{t}_e$. Hence, $y^{-1}\dot{x}$ normalizes $\mathfrak{t}_e$ and it acts on $\mathfrak{t}_e$ in the same way as x. So $W_e \subseteq W^J$. $\square$

Note that W_e leaves $\Phi_e \subset E_e$ invariant; hence it acts on bases for Φ_e, or equivalently, on the chambers of the hyperplane arrangement $\mathscr{A}_e$, as described by Lemma 11. Let W_e° denote the Weyl group of the reductive part $\mathfrak{c}_e$ of $\mathfrak{g}_e$, so W_e° is the subgroup of $\mathrm{GL}(E_e)$ generated by the reflections in the hyperplanes orthogonal to the simple roots Δ_e° of the root system Φ_e° of $\mathfrak{c}_e$. Let Z_e denote the stabilizer in W_e of the dominant chamber $\{\alpha \in E_e \mid (\alpha, \beta) > 0 \text{ for all } \beta \in \Delta_e^\circ\}$.

LEMMA 15. *We have* $W_e = Z_e \ltimes W_e^\circ$ *and* $Z_e \cong C_e/C_e^\circ Z_{C_e}(\mathfrak{t}_e)$, *a quotient of the component group* $C_e/C_e^\circ \cong G_e/G_e^\circ$.

Proof. Note that $W_e^\circ = N_{C_e^\circ}(\mathfrak{t}_e)/Z_{C_e^\circ}(\mathfrak{t}_e) \cong N_{C_e^\circ}(\mathfrak{t}_e)Z_{C_e}(\mathfrak{t}_e)/Z_{C_e}(\mathfrak{t}_e)$. Hence, recalling that $W_e = N_{C_e}(\mathfrak{t}_e)/Z_{C_e}(\mathfrak{t}_e)$, we see that the reflection group W_e° is a normal subgroup of W_e. Now one can see that $W_e = Z_e \ltimes W_e^\circ$; see [**9**, Lemma 2]. Moreover, we have shown that

$$Z_e \cong W_e/W_e^\circ \cong N_{C_e}(\mathfrak{t}_e)/N_{C_e^\circ}(\mathfrak{t}_e)Z_{C_e}(\mathfrak{t}_e).$$

Now consider the natural map $N_{C_e}(\mathfrak{t}_e) \to C_e/C_e^\circ Z_{C_e}(\mathfrak{t}_e)$. It is surjective because for every $x \in C_e$ there exists $y \in C_e^\circ$ with $x \cdot \mathfrak{t}_e = y \cdot \mathfrak{t}_e$. Its kernel is $N_{C_e^\circ}(\mathfrak{t}_e)Z_{C_e}(\mathfrak{t}_e)$. Hence it induces an isomorphism between Z_e and $C_e/C_e^\circ Z_{C_e}(\mathfrak{t}_e)$. $\square$

It follows from this and Lemma 11 that Z_e has $|\mathscr{K}_J|$ orbits on the set of chambers of the arrangement $\mathscr{A}_e$ that are contained in the dominant chamber

$$\{\alpha \in E_e \mid (\alpha, \beta) > 0 \text{ for all } \beta \in \Delta_e^\circ\},$$

and each orbit is regular. One can easily read off the structure of the group Z_e from the tables in [**9**] and [**4**]. For $\mathfrak{g}$ simple, the group Z_e is trivial, except in the following cases:

(i) $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ and λ has $k > 0$ distinct even parts of even multiplicity, in which case $Z_e \cong S_2 \times \ldots \times S_2$ (k times);
(ii) $\mathfrak{g} = \mathfrak{so}_N(\mathbb{C})$, at least one part of λ has odd multiplicity, and λ has $k > 0$ distinct odd parts of even multiplicity, in which case $Z_e \cong S_2 \times \ldots \times S_2$ (k times);
(iii) $\mathfrak{g} = \mathfrak{so}_N(\mathbb{C})$, all parts of λ are of even multiplicity, and λ has $k > 1$ distinct odd parts, in which case $Z_e \cong S_2 \times \ldots \times S_2$ ($(k-1)$ times);
(iv) $\mathfrak{g} = F_4$ and e has Bala–Carter label $\widetilde{A}_1$, A_2 or B_2, in which case $Z_e \cong S_2$;
(v) $\mathfrak{g} = E_6$ and e has Bala–Carter label A_2, in which case $Z_e \cong S_2$;
(vi) $\mathfrak{g} = E_6$ and e has Bala–Carter label $D_4(a_1)$, in which case $Z_e \cong S_3$;

(vii) $\mathfrak{g} = E_7$ and e has Bala–Carter label A_2, $A_2 + A_1$, $D_4(a_1) + A_1$, $A_3 + A_2$, A_4, $A_4 + A_1$, $D_5(a_1)$ or $E_6(a_1)$, in which case $Z_e \cong S_2$;

(viii) $\mathfrak{g} = E_7$ and e has Bala–Carter label $D_4(a_1)$, in which case $Z_e \cong S_3$;

(ix) $\mathfrak{g} = E_8$ and e has Bala–Carter label A_2, $A_2 + A_1$, $2A_2$, $A_3 + A_2$, A_4, $D_4(a_1) + A_2$, $A_4 + A_1$, $D_5(a_1)$, $A_4 + 2A_1$, $D_4 + A_2$, $D_6(a_2)$, $D_6(a_1)$, $E_6(a_1)$, $D_5 + A_2$, $D_7(a_2)$, $E_6(a_1) + A_1$ or $D_7(a_1)$, in which case $Z_e \cong S_2$;

(x) $\mathfrak{g} = E_8$ and e has Bala–Carter label $D_4(a_1)$ or $D_4(a_1) + A_1$, in which case $Z_e \cong S_3$.

In (i)–(iii), when $\mathfrak{g}$ is classical, the partition $\lambda = (1^{m_1} 2^{m_2} \dots)$ denotes the Jordan type of e in its natural representation.

Finally in this section, we wish to say a little more about the dimensions of the *root spaces* $\mathfrak{g}_e(\alpha, i)$ of $\mathfrak{g}_e$ for $\alpha \in \Phi_e$. Note that $\dim \mathfrak{g}_e(\alpha, i)$ is the same as the multiplicity $m(\alpha, i)$ from the proof of Lemma 13. By the definition of $\mathfrak{c}_e$, $m(\alpha, 0)$ is 1 or 0 according to whether $\alpha \in \Phi_e^\circ$ or not. The root multiplicities $m(\alpha, i)$ for $i > 0$ can often be greater than 1, and in fact can be arbitrarily large for symplectic and orthogonal Lie algebras. For $\mathfrak{g} = \mathfrak{sl}_n$, the multiplicities $m(\alpha, i)$ are always 1, and explicit calculations as described in the next paragraph show that $m(\alpha, i)$ is always at most 3 for G simple of exceptional type. In general, the root space $\mathfrak{g}_e(\alpha, i)$ need not be a subalgebra of $\mathfrak{g}_e$.

Let us explain exactly how to compute the root multiplicities $m(\alpha, i)$ from the root system of G. Let $I = \{1, \dots, r\} \setminus J$, so that as in the previous section $\{\alpha_i^J \mid i \in I\}$ is the standard base for the restricted root system Φ^J. Of course, the restriction α^J of a root $\alpha = \sum_{i=1}^r a_i \alpha_i$ is simply $\sum_{i \in I} a_i \alpha_i^J$, so it is easy to write down the set Φ^J explicitly given the root system of $\mathfrak{g}$. Since L centralizes $\mathfrak{t}_e$, it does no harm to conjugate by an element of L to assume that the distinguished $\mathfrak{sl}_2$-triple (e, h, f) in $[\mathfrak{l}, \mathfrak{l}]$ is in standard form, so that the values $\alpha_j(h)$ for $j \in J$ are all either 0 or 2 as can be read off from the labelled diagram for the distinguished nilpotent $e \in [\mathfrak{l}, \mathfrak{l}]$ from [**4**]. Solving some linear equations, one can then uniquely determine the other values $\alpha_i(h)$ for $i \in I$, using the fact that h is orthogonal to $\mathfrak{t}_e$. Hence, we can compute all the integers $\beta(h)$ for $\beta \in \Phi$. Now take $\alpha \in \Phi_e$. The formal character of the $\mathfrak{s}$-module arising from the α-weight space of $\mathfrak{g}$ with respect to $\mathfrak{t}_e$ is then

$$\sum_{\beta \in \Phi : \beta^J = \alpha} x^{\beta(h)}.$$

By $\mathfrak{sl}_2$-theory, this can be written uniquely as

$$\sum_{i \geqslant 0} m(\alpha, i)(x^i + x^{i-2} + \dots + x^{-i})$$

for integers $m(\alpha, i) \geqslant 0$. These are the desired multiplicities. For exceptional groups, this procedure is particularly effective, and we have implemented it in GAP [**8**] to quickly compute all root multiplicities in all cases, though there does not seem to be a compact way to present this information here. For classical groups, there is a different approach based on diagrams in the plane called *pyramids*; this is described in Sections 6–8.

Example 16. Take $G = E_7$ and e with Bala–Carter label $A_3 + A_2$. Continuing with the notation from Example 12, we see that the values of $\alpha_i(h)$ for $i = 1, \dots, 7$ are given by the labelled Dynkin diagram

$$\begin{array}{cccccc} 2 & 2 & -5 & 2 & 2 & 2 \\ & & 0 & & & \end{array}$$

From this, one can compute the root multiplicities $m(\alpha, i)$ by the method just explained. To record these, we list for every $\alpha \in \Phi_e^+$ the sequence made up of the positive integers $m(\alpha, i)$ for all $i \geqslant 0$: $\alpha_1^J : 0$; $\alpha_2^J : 1, 3, 5$; $\alpha_1^J + \alpha_2^J : 1, 3, 5$; $\alpha_1^J + 2\alpha_2^J : 2, 2, 4, 6$; $\alpha_1^J + 3\alpha_2^J : 3$; $2\alpha_1^J + 3\alpha_2^J : 3$; $2\alpha_1^J + 4\alpha_2^J : 2$. The reductive part C_e of the centralizer is of type $A_1 + T_1$, and $\Delta_e^\circ = \{\alpha_1^J\}$. Hence, the fundamental chamber is the right half plane in Figure 1 in Example 12, and $W_e^\circ \cong S_2$

is generated by the reflection in the vertical axis. The group $Z_e \cong S_2$ is generated by the reflection in the horizontal axis.

4. *Good gradings*

Continue with notation as in the previous section. In particular, we have fixed bases $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ for Φ and $\Delta_e = \{\alpha_i^J \mid i \in I\}$ for Φ_e. We often now represent an element $c \in \mathfrak{t}$ as a tuple $(c_1, \ldots, c_r)$ of complex numbers, where $c_i = \alpha_i(c)$. Of course, we think of this as a labelling of the vertices of the Dynkin diagram. By a *grading* of $\mathfrak{g}$, we always mean an $\mathbb{R}$-grading

$$\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. We say that the grading Γ is *compatible with* $\mathfrak{t}$ if $\mathfrak{t} \subseteq \mathfrak{g}_0$. Since every derivation of $\mathfrak{g}$ is inner, there exists a unique semisimple element $c \in \mathfrak{g}$ defining Γ, that is, so that $\mathfrak{g}_j$ is the j-eigenspace of $\operatorname{ad} c$. The grading is compatible with $\mathfrak{t}$ if and only if this element belongs to $\mathfrak{t}$. In this way, gradings of $\mathfrak{g}$ that are compatible with $\mathfrak{t}$ are parameterized by labelled Dynkin diagrams $(c_1, \ldots, c_r)$ with all labels $c_i \in \mathbb{R}$. Every semisimple element of $\mathfrak{g}$ is G-conjugate to an element of $\mathfrak{t}$, so every grading is G-conjugate to a grading that is compatible with $\mathfrak{t}$. Finally, two elements of $\mathfrak{t}$ are G-conjugate if and only if they are W-conjugate; hence every grading of $\mathfrak{g}$ is G-conjugate to a unique grading that is compatible with $\mathfrak{t}$ and whose labelled Dynkin diagram $(c_1, \ldots, c_r)$ has all labels $c_i \in \mathbb{R}_{\geqslant 0}$. We call this labelled diagram the *characteristic* of the grading.

Now assume that $\Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$ is a good grading for e as defined in the introduction. The proof of [**6**, Theorem 1.3] shows that $\operatorname{ad} e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective if and only if $\operatorname{ad} e : \mathfrak{g}_{-j-2} \to \mathfrak{g}_{-j}$ is injective. Hence, the conditions that $\operatorname{ad} e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for all $j \leqslant -1$ and that $\operatorname{ad} e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for all $j \geqslant -1$ in the definition of good grading are in fact equivalent. So, Γ is a good grading for $e \in \mathfrak{g}_2$ if and only if $\mathfrak{g}_e \subseteq \bigoplus_{j > -1} \mathfrak{g}_j$.

LEMMA 17. *Let Γ be a grading of $\mathfrak{g}$ with $e \in \mathfrak{g}_2$. Then we have* $\dim \mathfrak{g}_e \geqslant \sum_{-1 \leqslant j < 1} \dim \mathfrak{g}_j$ *with equality if and only if Γ is a good grading for e.*

Proof. The discussion before the statement of the lemma implies that it can be proved in the same way as [**6**, Corollary 1.3]. □

By a *good characteristic*, we mean the characteristic of a good grading for e. By the proof of [**6**, Theorem 1.2], a good characteristic $(c_1, \ldots, c_r)$ always has the property that $0 \leqslant c_i \leqslant 2$ for all $i = 1, \ldots, r$. We should observe that the original good grading Γ for e can be recovered from its characteristic uniquely up to conjugacy by G_e; this means that good characteristics parameterize G_e-conjugacy classes of good gradings for e. To see this, suppose that Γ and Γ' are two good gradings for e with the same characteristic. There certainly exists $y \in G$ such that $y \cdot \Gamma' = \Gamma$. So Γ is good both for e and for $y \cdot e$. Let G_0 be the set of all elements of G that preserve the grading Γ, that is, $G_0 = \{x \in G : x \cdot \mathfrak{g}_i = \mathfrak{g}_i \text{ for all } i \in \mathbb{R}\}$. Lemma 18 below implies that $y \cdot e = z \cdot e$ for some $z \in G_0$. But then $z^{-1}y \cdot \Gamma' = \Gamma$ too, and $z^{-1}y \in G_e$, as required.

LEMMA 18. *If Γ is a good grading, the set of all elements $e \in \mathfrak{g}_2$ such that Γ is a good grading for e is a dense open orbit for the action of G_0 on $\mathfrak{g}_2$.*

Proof. Suppose that Γ is a good grading for e and for e'. We have $[e, \mathfrak{g}_0] = \mathfrak{g}_2$. Hence, $\dim G_0 \cdot e = \dim \mathfrak{g}_2$, and $G_0 \cdot e$ is dense open in $\mathfrak{g}_2$. So is $G_0 \cdot e'$, so $G_0 \cdot e$ and $G_0 \cdot e'$ have non-empty intersection. Hence, $G_0 \cdot e = G_0 \cdot e'$. □

For the next lemma, we note that E_e is the $\mathbb{R}$-form for $\mathfrak{t}_e$ consisting of all $p \in \mathfrak{t}_e$ such that the eigenvalues of $\operatorname{ad} p$ on $\mathfrak{g}$ are real. Also recall that the element h from our fixed $\mathfrak{sl}_2$-triple (e, h, f) belongs to $\mathfrak{t}$, since $\mathfrak{t}$ was chosen originally to lie in $\mathfrak{c} = \mathfrak{z}_{\mathfrak{g}}(h)$.

LEMMA 19. *Every G_e-conjugacy class of good gradings for e has a representative Γ that is compatible with $\mathfrak{t}$. Moreover, for any such Γ, we have $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$, and the element $c \in \mathfrak{g}$ defining the grading Γ is of the form $c = h + p$ for some point $p \in E_e$.*

Proof. Let Γ be any good grading for e. As in [**6**, Lemma 1.1], there exists an $\mathfrak{sl}_2$-triple (e, h', f') with $h' \in \mathfrak{g}_0$ and $f' \in \mathfrak{g}_{-2}$. This is conjugate to our fixed $\mathfrak{sl}_2$-triple $\mathfrak{s} = (e, h, f)$ by an element x of G_e°. On replacing Γ by $x \cdot \Gamma$ if necessary, we may therefore assume that $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$ already. Let c be the semisimple element of $\mathfrak{g}$ defining the grading Γ. Since $[h, e] = [c, e] = 2e$, the element $p = c - h$ centralizes e. Since $h \in \mathfrak{g}_0$, we have $[h, c] = 0$, so p is a semisimple element of $\mathfrak{c}_e$. Recalling that $\mathfrak{t}_e$ is a Cartan subalgebra of $\mathfrak{c}_e$, we can therefore conjugate once more by an element of C_e° to reduce to the situation where $p \in \mathfrak{t}_e$. But then $c = h + p$ belongs to $\mathfrak{t}$, and the grading Γ is compatible with $\mathfrak{t}$ as required.

Now suppose that Γ is any good grading for e that is compatible with $\mathfrak{t}$. Let c be the element of $\mathfrak{t}$ defining the grading, so that $[c, e] = 2e$ and $[c, h] = 0$, that is, $h \in \mathfrak{g}_0$. This implies that $p = c - h$ centralizes e and h, and hence also f, so $[c, f] = [h, f] = -2f$ and $f \in \mathfrak{g}_{-2}$. Finally, observe that p belongs to $\mathfrak{t}_e$, and hence to E_e since Γ is an $\mathbb{R}$-grading. □

Given any $p \in E_e$, let $\Gamma(p)$ denote the grading of $\mathfrak{g}$ defined by the semisimple element $h + p$. For example, $\Gamma(0)$ is the Dynkin grading. In general, the grading $\Gamma(p)$ is certainly compatible with $\mathfrak{t}$ and the element e is in degree 2. However, it need not be a good grading for e.

THEOREM 20. *For $p \in E_e$, the grading $\Gamma(p)$ is a good grading for e if and only if $|\alpha(p)| < d(\alpha)$ for all $\alpha \in \Phi_e^+$, where $d(\alpha) = 1 + \min\{i \geqslant 0 \mid m(\alpha, i) \neq 0\}$, that is, the minimal dimension of an irreducible $\mathfrak{s}$-submodule of the α-weight space of $\mathfrak{g}$ with respect to $\mathfrak{t}_e$.*

Proof. Recall that $\Gamma(p) : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}_j$ is a good grading for e if and only if $\mathfrak{g}_e \subseteq \bigoplus_{j > -1} \mathfrak{g}_j$. Let $\mathfrak{g} = \bigoplus_{\alpha \in \Phi_e \cup \{0\}} \bigoplus_{i \geqslant 0} m(\alpha, i) L(\alpha, i)$ be the decomposition of $\mathfrak{g}$ as a $\mathfrak{t}_e \oplus \mathfrak{s}$-module, as in the proof of Lemma 13. Note that $h + p$ acts on the highest weight vector of $L(\alpha, i)$ as the scalar $\alpha(p) + i$. Putting these things together, we find that $\Gamma(p)$ is a good grading for e if and only if $\alpha(p) + i > -1$ whenever $m(\alpha, i) \neq 0$. Since $\Phi_e = \Phi_e^+ \sqcup (-\Phi_e^+)$, the theorem follows easily. □

Let $\mathscr{P}_e$ denote the set of all $p \in E_e$ such that $\Gamma(p)$ is a good grading for e. Since Φ_e^+ spans E_e^*, Theorem 20 shows in particular that $\mathscr{P}_e$ is an open convex polytope in the real vector space E_e. It can be computed explicitly from information about the root multiplicities $m(\alpha, i)$; see the discussion before Example 16. We call it the *good grading polytope* corresponding to e. By Lemma 19, the map $p \mapsto \Gamma(p)$ gives a bijection between $\mathscr{P}_e$ and the set of all good gradings for e that are compatible with $\mathfrak{t}$. The next theorem describes exactly when the good gradings $\Gamma(p)$ and $\Gamma(p')$ for e are conjugate, for points $p, p' \in \mathscr{P}_e$. Recall that this is so if and only if they have the same characteristic.

THEOREM 21. *For $p, p' \in \mathscr{P}_e$, the good gradings $\Gamma(p)$ and $\Gamma(p')$ are G-conjugate if and only if p and p' are W_e-conjugate.*

Proof. Suppose that the good gradings $\Gamma(p) : \mathfrak{g} = \bigoplus_{i \in \mathbb{R}} \mathfrak{g}_i$ and $\Gamma(p') : \mathfrak{g} = \bigoplus_{j \in \mathbb{R}} \mathfrak{g}'_j$ are G-conjugate. Since they are both good gradings for e, they are already conjugate under the centralizer G_e, as we explained earlier. So we can find $x \in G_e$ such that $x \cdot (h + p) = h + p'$. Since $h + p'$ lies in both $\mathfrak{t}$ and in $x \cdot \mathfrak{t}$, it centralizes $\mathfrak{t}_e$ and $x \cdot \mathfrak{t}_e$, so both $\mathfrak{t}_e$ and $x \cdot \mathfrak{t}_e$ lie in $\mathfrak{g}'_0$.

They also clearly both lie in $\mathfrak{g}_e$; hence $\mathfrak{t}_e$ and $x \cdot \mathfrak{t}_e$ are Cartan subalgebras of $\mathfrak{g}_e \cap \mathfrak{g}'_0$. Let G'_0 be the subgroup of G consisting of all elements that preserve the grading $\Gamma(p')$. Then we deduce that there exists an element $y \in G_e \cap G'_0$ such that $y \cdot \mathfrak{t}_e = x \cdot \mathfrak{t}_e$. Hence, $y^{-1}x \cdot (h+p) = h+p'$ and $y^{-1}x \in N_{G_e}(\mathfrak{t}_e)$. Since $y^{-1}x$ normalizes $\mathfrak{t}_e$, it normalizes $\mathfrak{l}$, and hence $[\mathfrak{l}, \mathfrak{l}]$. So, $y^{-1}x \cdot h = h'$ for some $h' \in [\mathfrak{l}, \mathfrak{l}]$, and $h + p' = y^{-1}x \cdot (h+p) = h' + y^{-1}x \cdot p$. This shows that $h = h'$ and $y^{-1}x \cdot p = p'$ already. Hence, p and p' are W_e-conjugate.

Conversely, suppose that p and p' are W_e-conjugate. Then, recalling that

$$W_e = N_{C_e}(\mathfrak{t}_e)/Z_{C_e}(\mathfrak{t}_e),$$

we can find $x \in N_{C_e}(\mathfrak{t}_e)$ with $x \cdot p = p'$. Since x lies in C_e, it centralizes h, so $x \cdot (h+p) = h+p'$. Hence, $\Gamma(p)$ and $\Gamma(p')$ are conjugate. $\square$

We finally introduce the *affine hyperplanes*

$$H_{\alpha,k} = \{p \in E_e \mid \alpha(p) = k\}$$

for each $\alpha \in \Phi_e^+$ and $k \in \mathbb{Z}$. The significance of these will be discussed in detail in the next section. We just want to point out here that the *integral* good gradings for e that are compatible with $\mathfrak{t}$ are parameterized by the points $p \in \mathscr{P}_e$ such that $\alpha(p) \in \mathbb{Z}$ for all $\alpha \in \Phi_e^+$. In other words, $\Gamma(p)$ is an integral grading if and only if p lies on the same number of the affine hyperplanes $H_{\alpha,k}$ for $\alpha \in \Phi_e^+$ and $k \in \mathbb{Z}$ as the origin (which corresponds to the Dynkin grading). Actually, it is often the case that the Dynkin grading is the only integral good grading, as is well explained by [**6**, Corollary 1.1].

EXAMPLE 22. Continue with $G = E_7$ and e having Bala–Carter label $A_3 + A_2$, with notation as in Examples 12 and 16. In Example 16, we computed all the root multiplicities $m(\alpha, i)$. Hence, according to Theorem 20, the good grading polytope $\mathscr{P}_e$ is the subspace of E_e defined by the inequalities

$$|\alpha_1^J(p)| < 1, \quad |\alpha_2^J(p)| < 2, \quad |(\alpha_1^J + \alpha_2^J)(p)| < 2, \quad |(\alpha_1^J + 2\alpha_2^J)(p)| < 3,$$
$$|(\alpha_1^J + 3\alpha_2^J)(p)| < 4, \quad |(2\alpha_1^J + 3\alpha_2^J)(p)| < 4, \quad |(2\alpha_1^J + 4\alpha_2^J)(p)| < 3.$$

These are equivalent just to the inequalities $|\alpha_1^J(p)| < 1$ and $|(\alpha_1^J + 2\alpha_2^J)(p)| < \frac{3}{2}$, so the good grading polytope can be represented as the interior of the rectangle in Figure 2 (on the same

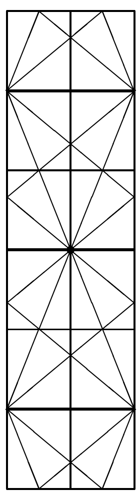


FIGURE 2

axes as in Example 12 but with a different scale). We have also indicated on the diagram the affine hyperplanes that intersect with $\mathscr{P}_e$. From this, we see that no other point of $\mathscr{P}_e$ lies on as many affine hyperplanes as the origin. So the only integral good grading for e is the Dynkin grading, which is consistent with the classification of integral good gradings from [**6**] in this case.

5. *Alcoves and adjacencies*

Recall from the introduction that a pair $\Gamma : \mathfrak{g} = \bigoplus_{i\in\mathbb{R}} \mathfrak{g}_i$ and $\Gamma' : \mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}'_j$ of good gradings for e are *adjacent* if $\mathfrak{g} = \bigoplus_{i^- \leqslant j \leqslant i^+} \mathfrak{g}_i \cap \mathfrak{g}'_j$. Note this means in particular that the gradings Γ and Γ' are compatible with each other, that is, the semisimple elements of $\mathfrak{g}$ that define the gradings Γ and Γ' commute. Moreover, $\bigoplus_{i,j\geqslant 0} \mathfrak{g}_i \cap \mathfrak{g}'_j$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0 \cap \mathfrak{g}'_0$. So we can find an element $x \in G$ such that $x \cdot \mathfrak{t} \subseteq \mathfrak{g}_0 \cap \mathfrak{g}'_0$ and $x \cdot \mathfrak{b} \subseteq \bigoplus_{i,j\geqslant 0} \mathfrak{g}_i \cap \mathfrak{g}'_j$. The characteristics $(c_1, \dots, c_r)$ and $(c'_1, \dots, c'_r)$ of the gradings Γ and Γ' can then be read off simultaneously from the equations $x \cdot \mathfrak{g}_{\alpha_i} \subseteq \mathfrak{g}_{c_i} \cap \mathfrak{g}_{c'_i}$ for all $i = 1, \dots, r$.

We say that two good characteristics $(c_1, \dots, c_r)$ and $(c'_1, \dots, c'_r)$ are *adjacent* if they are the characteristics of a pair of adjacent good gradings Γ and Γ' for e. If we are given just a pair $(c_1, \dots, c_r)$ and $(c'_1, \dots, c'_r)$ of adjacent good characteristics, we can recover the pair Γ and Γ' of adjacent good gradings for e from which they were defined uniquely up to conjugation by G_e. To see this, the previous paragraph implies that Γ and Γ' can be obtained by simultaneously conjugating the two gradings defined by declaring that each $\mathfrak{g}_{\pm\alpha_i}$ is in degree $\pm c_i$ or in degree $\pm c'_i$, respectively, by some element $x \in G$. So we just need to show that if Λ and Λ' are another pair of good gradings for e with $y \cdot \Lambda = \Gamma$ and $y \cdot \Lambda' = \Gamma'$ for some $y \in G$, then in fact Λ and Λ' are already conjugate to Γ and Γ' by an element of G_e. For this, note that Γ and Γ' are good gradings both for e and for $y \cdot e$. So by Lemma 23 below, there exists an element $z \in G$ preserving both gradings Γ and Γ' with $z \cdot e = y \cdot e$. But then $z^{-1}y \cdot \Lambda = \Gamma$ and $z^{-1}y \cdot \Lambda' = \Gamma'$, and $z^{-1}y \in G_e$, as required.

LEMMA 23. *Let Γ and Γ' be adjacent good gradings for e. Let G_0 and G'_0 be the subgroups of G consisting of all elements that preserve the gradings Γ and Γ', respectively. The set of all elements e' of $\mathfrak{g}_2 \cap \mathfrak{g}'_2$ such that both Γ and Γ' are good gradings for e' is a dense open orbit for the action of the group $G_0 \cap G'_0$.*

Proof. Let $\mathfrak{g}_{<j}$ denote $\bigoplus_{i<j} \mathfrak{g}_i$. Define $\mathfrak{g}_{>j}$, $\mathfrak{g}'_{<j}$ and $\mathfrak{g}'_{>j}$ similarly. Since the map

$$\operatorname{ad} e : \mathfrak{g}_0 \to \mathfrak{g}_2$$

is surjective and preserves the direct sum decompositions

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{g}'_{<0}) \oplus (\mathfrak{g}_0 \cap \mathfrak{g}'_0) \oplus (\mathfrak{g}_0 \cap \mathfrak{g}'_{>0}) \quad \text{and} \quad \mathfrak{g}_2 = (\mathfrak{g}_2 \cap \mathfrak{g}'_{<2}) \oplus (\mathfrak{g}_2 \cap \mathfrak{g}'_2) \oplus (\mathfrak{g}_2 \cap \mathfrak{g}'_{>2}),$$

we have $[\mathfrak{g}_0 \cap \mathfrak{g}'_0, e] = \mathfrak{g}_2 \cap \mathfrak{g}'_2$. Now argue as in the proof of Lemma 18. □

This shows that adjacent good characteristics parameterize G_e-conjugacy classes of adjacent good gradings for e. In order to classify all adjacent good gradings, and hence all adjacent good characteristics, in terms of the good grading polytope we use the following lemma.

LEMMA 24. *Let Γ and Γ' be adjacent good gradings for e. Then there exists $x \in G_e^\circ$ such that both $x \cdot \Gamma$ and $x \cdot \Gamma'$ are compatible with $\mathfrak{t}$.*

Proof. Note that $e \in \mathfrak{g}_2 \cap \mathfrak{g}'_2$ by the definition of good grading. Arguing as in the proof of [**6**, Lemma 1.1], working with the bigrading

$$\mathfrak{g} = \bigoplus_{i,j\in\mathbb{Z}} \mathfrak{g}_i \cap \mathfrak{g}'_j$$

instead of the grading used there, one shows that there exists an $\mathfrak{sl}_2$-triple (e, h', f') with $h' \in \mathfrak{g}_0 \cap \mathfrak{g}'_0$ and $f' \in \mathfrak{g}_{-2} \cap \mathfrak{g}'_{-2}$. Conjugating by an element of G_e° if necessary, we may assume that in fact $h' = h$ and $f' = f$, that is, $h \in \mathfrak{g}_0 \cap \mathfrak{g}'_0$ and $f \in \mathfrak{g}_{-2} \cap \mathfrak{g}'_{-2}$ already. Now argue as in the proof of Lemma 19 to show that $\mathfrak{t} \subseteq \mathfrak{g}_0 \cap \mathfrak{g}'_0$. □

Assume that Γ and Γ' are adjacent good gradings for e. In view of Lemma 24, we can conjugate by an element of G_e° if necessary to assume that Γ and Γ' are both compatible with $\mathfrak{t}$. Then, $\Gamma = \Gamma(p)$ and $\Gamma' = \Gamma(p')$ for points $p, p' \in \mathscr{P}_e$. In this way, the problem of determining all conjugacy classes of pairs of adjacent good gradings for e reduces to describing exactly when $\Gamma(p)$ and $\Gamma(p')$ are adjacent. To do this, recall the affine hyperplanes $\{H_{\alpha,k} \mid \alpha \in \Phi_e^+, k \in \mathbb{Z}\}$ introduced at the end of the previous section. We refer to the connected components of

$$E_e \setminus \bigcup_{\substack{\alpha\in\Phi_e^+\\ k\in\mathbb{Z}}} H_{\alpha,k}$$

as *alcoves*. Note that the closure of $\mathscr{P}_e$ is the union of the closures of finitely many alcoves.

THEOREM 25. *Let $p, p' \in \mathscr{P}_e$. Then, $\Gamma(p)$ and $\Gamma(p')$ are adjacent if and only if p and p' belong to the closure of the same alcove.*

Proof. By definition, $\mathfrak{g}_\alpha$ lies in the degree $\alpha(h) + \alpha(p)$ piece of the grading $\Gamma(p)$, for each $\alpha \in \Phi$. Since $\alpha(h)$ is always an integer, it follows that $\Gamma(p)$ and $\Gamma(p')$ are adjacent if and only if $\alpha(p)^- \leqslant \alpha(p') \leqslant \alpha(p)^+$ for all $\alpha \in \Phi$. Equivalently, p and p' belong to the closure of the same alcove. □

Theorems 20 and 25 combine to prove Theorem 2 from the introduction. In the remainder of the section, we want to explain the remaining ingredients needed to deduce Theorem 1 from it, as we outlined in the introduction. The first step is accomplished by the following lemma. Recall from the introduction that $\langle\, , \rangle$ is the skew-symmetric bilinear form defined by $\langle x, y\rangle = ([x, y], e)$.

LEMMA 26. *Let $\Gamma : \mathfrak{g} = \bigoplus_{i\in\mathbb{R}} \mathfrak{g}_i$ and $\Gamma' : \mathfrak{g} = \bigoplus_{j\in\mathbb{R}} \mathfrak{g}_j$ be adjacent good gradings for e. Then there exist Lagrangian subspaces $\mathfrak{k}$ of $\mathfrak{g}_{-1}$ and $\mathfrak{k}'$ of $\mathfrak{g}'_{-1}$ (both with respect to the form $\langle\, , \rangle$) such that*

$$\mathfrak{k} \oplus \bigoplus_{i<-1} \mathfrak{g}_i = \mathfrak{k}' \oplus \bigoplus_{j<-1} \mathfrak{g}'_j.$$

Proof. In the notation from the proof of Lemma 23, we have

$$\begin{aligned}
\mathfrak{g}_{-1} &= ((\mathfrak{g}_{-1} \cap \mathfrak{g}'_{<-1}) \oplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{>-1})) \perp (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}),\\
\mathfrak{g}'_{-1} &= ((\mathfrak{g}_{<-1} \cap \mathfrak{g}'_{-1}) \oplus (\mathfrak{g}_{>-1} \cap \mathfrak{g}'_{-1})) \perp (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}),
\end{aligned}$$

where the $\perp$ are with respect to the form $\langle\, , \rangle$. Hence, the restriction of $\langle\, , \rangle$ to $\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}$ is non-degenerate. Let $\mathfrak{k}''$ be a Lagrangian subspace of $\mathfrak{g}_{-1} \cap \mathfrak{g}'_{-1}$. Then set $\mathfrak{k} = \mathfrak{k}'' \oplus (\mathfrak{g}_{-1} \cap \mathfrak{g}'_{<-1})$ and $\mathfrak{k}' = \mathfrak{k}'' \oplus (\mathfrak{g}_{<-1} \cap \mathfrak{g}'_{-1})$. This does the job in view of the definition of adjacency. □

It just remains to indicate how to adapt the argument of Gan and Ginzburg [**7**] to our slightly more general setting. For an integer $d \geqslant 1$, we define a *d-good grading* for e to be a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with $e \in \mathfrak{g}_d$, such that $\operatorname{ad} e : \mathfrak{g}_j \to \mathfrak{g}_{j+d}$ is injective for $j \leqslant -\frac{1}{2}d$ and surjective for $j \geqslant -\frac{1}{2}d$. Given any good $\mathbb{R}$-grading Γ for e in our old sense, it is easy to see that one can always replace Γ by a good $\mathbb{Q}$-grading without changing $e \in \mathfrak{g}_2$ or either of the spaces $\mathfrak{g}_{-1}$ or $\bigoplus_{j<-1} \mathfrak{g}_j$, which is all that matters for the construction of finite W-algebras. In turn, since $\mathfrak{g}$ is finite dimensional, any good $\mathbb{Q}$-grading for $e \in \mathfrak{g}_2$ can be scaled by a sufficiently large integer d so that it becomes a $2d$-good grading for $e \in \mathfrak{g}_{2d}$ in the new sense. This reduces to the situation that $\Gamma : \mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ is a $2d$-good grading for e.

Next let $\mathfrak{k}$ be any isotropic subspace of $\mathfrak{g}_{-d}$, and let $\mathfrak{k}^\perp \subseteq \mathfrak{g}_{-d}$ be its annihilator with respect to the form $\langle\,,\rangle$. Let $\mathfrak{m} = \mathfrak{k} \oplus \bigoplus_{i<-d} \mathfrak{g}_i$ and $\mathfrak{n} = \mathfrak{k}^\perp \oplus \bigoplus_{j<-d} \mathfrak{g}_j$. Note that $\chi : \mathfrak{m} \to \mathbb{C}$, $x \mapsto (x, e)$ is a representation of $\mathfrak{m}$. Set $Q_\mathfrak{k} = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi = U(\mathfrak{g})/I_\mathfrak{k}$, where $I_\mathfrak{k}$ is the left ideal of $U(\mathfrak{g})$ generated by all $\{x - \chi(x) \mid x \in \mathfrak{m}\}$. Since $I_\mathfrak{k}$ is stable under the adjoint action of $\mathfrak{n}$, we get an induced action of $\mathfrak{n}$ on $Q_\mathfrak{k}$, by $x \cdot (u \otimes 1_\chi) = [x, u] \otimes 1_\chi$ for $x \in \mathfrak{n}$ and $u \in U(\mathfrak{g})$. Let $H_\mathfrak{k}$ be the space $Q_\mathfrak{k}^\mathfrak{n}$ of $\mathfrak{n}$-fixed points with respect to this action. It has a well-defined algebra structure defined by $(u \otimes 1_\chi)(v \otimes 1_\chi) = (uv) \otimes 1_\chi$, for $u, v \in U(\mathfrak{g})$ such that $u \otimes 1_\chi, v \otimes 1_\chi \in Q_\mathfrak{k}^\mathfrak{n}$. Now we can formulate the slight generalization of Gan and Ginzburg's theorem that we need here.

THEOREM 27. *Let $\mathfrak{k} \subseteq \mathfrak{k}'$ be two isotropic subspaces of $\mathfrak{g}_{-d}$, and define the corresponding algebras $H_\mathfrak{k}$ and $H_{\mathfrak{k}'}$ as above. The natural map $Q_\mathfrak{k} \to Q_{\mathfrak{k}'}$ induced by the inclusion $\mathfrak{k} \hookrightarrow \mathfrak{k}'$ restricts to an algebra isomorphism $H_\mathfrak{k} \to H_{\mathfrak{k}'}$.*

If we take $\mathfrak{k} = \{0\}$ and $\mathfrak{k}'$ and $\mathfrak{k}''$ to be any two Lagrangian subspaces of $\mathfrak{g}_{-d}$, this theorem gives isomorphisms from $H_\mathfrak{k}$ to both $H_{\mathfrak{k}'}$ and $H_{\mathfrak{k}''}$. Composing one with the inverse of the other, we get a canonical isomorphism between $H_{\mathfrak{k}'}$ and $H_{\mathfrak{k}''}$. In turn, by the Frobenius reciprocity argument explained in the introduction of [**3**], $H_{\mathfrak{k}'}$ is naturally isomorphic to the finite W-algebra $H_{\chi'}$ from the introduction defined from the Lagrangian subspace $\mathfrak{k}'$, while $H_{\mathfrak{k}''}$ is naturally isomorphic to $H_{\chi''}$ defined from $\mathfrak{k}''$. In this way, we obtain the canonical isomorphism between $H_{\chi'}$ and $H_{\chi''}$ needed to prove Theorem 1.

The proof of Theorem 27 itself is almost exactly the same as the proof of the second part of [**7**, Theorem 4.1]. We just note here that one needs to replace the linear action ρ of $\mathbb{C}^\times$ on $\mathfrak{g}$ from [**7**] with one defined by $\rho(t)(x) = t^{d-j}x$, for $t \in \mathbb{C}^\times$ and $x \in \mathfrak{g}_j$. There is a corresponding Kazhdan filtration on $U(\mathfrak{g})$ as in [**7**, § 4]. We also note the identity

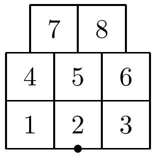
$$\mathfrak{m}^\perp = [\mathfrak{n}, e] \oplus \mathfrak{z}_\mathfrak{g}(f),$$

where $\mathfrak{m}^\perp$ is the annihilator of $\mathfrak{m}$ in $\mathfrak{g}$ with respect to the Killing form and (e, h, f) is an $\mathfrak{sl}_2$-triple with $f \in \mathfrak{g}_{-d}$; this is proved as in [**7**, (2.2)] using Lemma 17. Combining these things, one gets the analogue of [**7**, Lemma 2.1], which is the key lemma needed in the spectral sequence argument used to prove [**7**, Theorem 4.1].

6. *Good gradings for $\mathfrak{sl}_n(\mathbb{C})$*

In this section, we describe explicitly the restricted root systems and the good grading polytopes arising from $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Let V denote the natural n-dimensional $\mathfrak{g}$-module of column vectors, with standard basis $v_1, \ldots, v_n$. Also let $\mathfrak{t}$ be the standard Cartan subalgebra consisting of all diagonal matrices in $\mathfrak{g}$. Letting δ_i be the element of $\mathfrak{t}^*$ picking out the ith diagonal entry of a matrix in $\mathfrak{t}$, we find that the elements $\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n$ give a base Δ for the root system Φ.

Nilpotent orbits in $\mathfrak{g}$ are parameterized by partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots)$ of n. Fix such a partition λ throughout the section having a total of m non-zero parts. In order to write down an $\mathfrak{sl}_2$-triple corresponding to the partition λ explicitly, we first recall the definition of the *Dynkin pyramid* of shape λ, following [**6**]. This is a diagram consisting of n boxes each of size 2 units by 2 units drawn in the upper half of the xy-plane. By the coordinates of a box, we mean the coordinates of its midpoint. We will also talk about the *row number* of a box, meaning its y-coordinate, and the *column number* of a box, meaning its x-coordinate. Letting $r_i = 2i-1$ for short, we see that the Dynkin pyramid has λ_i boxes in row r_i centered in columns $1-\lambda_i, 3-\lambda_i, \dots, \lambda_i - 1$, for each $i = 1, \dots, m$. For example, here is the Dynkin pyramid of shape $\lambda = (3,3,2)$:



We fix once and for all a numbering $1, 2, \dots, n$ of the boxes of the Dynkin pyramid, and let $\mathrm{row}(i)$ and $\mathrm{col}(i)$ denote the row and column numbers of the ith box. Writing $e_{i,j}$ for the ij-matrix unit, let $e = \sum_{i,j} e_{i,j}$ summing over all $1 \leqslant i,j \leqslant n$ such that $\mathrm{row}(i) = \mathrm{row}(j)$ and $\mathrm{col}(i) = \mathrm{col}(j) + 2$. This is a nilpotent matrix of Jordan type λ. For example, taking $\lambda = (3,3,2)$ and numbering boxes as above, we have $e = e_{8,7} + e_{6,5} + e_{5,4} + e_{3,2} + e_{2,1}$. Also let $h = \sum_{i=1}^{n} \mathrm{col}(i) e_{i,i}$. There is then a unique element $f \in \mathfrak{g}$ such that (e,h,f) is an $\mathfrak{sl}_2$-triple.

With these choices, it is the case that $\mathfrak{t}$ is contained in $\mathfrak{c}$ and $\mathfrak{t}_e$ is a Cartan subalgebra of $\mathfrak{c}_e$, as was required in Section 3. Explicitly, $\mathfrak{t}_e$ consists of all matrices in $\mathfrak{t}$ such that the ith and jth diagonal entries are equal whenever $\mathrm{row}(i) = \mathrm{row}(j)$, and the real vector space E_e consists of all such matrices with entries in $\mathbb{R}$. Let $\varepsilon_i \in E_e^*$ be the function picking out the jth diagonal entry of a matrix in E_e, where j here is chosen so that the jth box of the Dynkin pyramid is in row r_i. Then, E_e^* is the $(m-1)$-dimensional real vector space spanned by $\varepsilon_1, \dots, \varepsilon_m$ subject to the relation $\sum_{i=1}^{m} \lambda_i \varepsilon_i = 0$. It is natural to identify E_e and E_e^* via the real inner product arising from the trace form on $\mathfrak{g}$, with respect to which $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}/\lambda_i$. We have

$$\Phi_e = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i,j \leqslant m, i \neq j\}.$$

Setting $p_i = \varepsilon_i(p)$, we see that any point $p \in E_e$ can be represented as a tuple $(p_1, \dots, p_m) \in \mathbb{R}^m$ with $\sum_{i=1}^{m} \lambda_i p_i = 0$. The values of $d(\alpha)$ from Theorem 20 can be determined using an explicit description of $\mathfrak{g}_e$ as in [**6**]: we have

$$d(\varepsilon_i - \varepsilon_j) = 1 + |\lambda_i - \lambda_j|$$

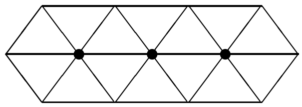
for all $1 \leqslant i,j \leqslant m$. Therefore, the good grading polytope $\mathscr{P}_e$ is the open subset of E_e consisting of all points $p = (p_1, \dots, p_m)$ such that

$$|p_i - p_j| < 1 + \lambda_i - \lambda_j$$

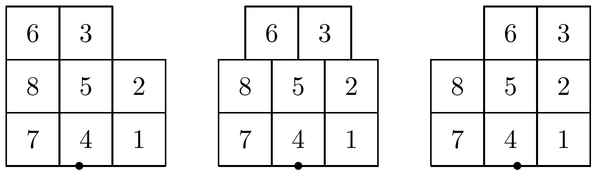
for all $1 \leqslant i < j \leqslant m$. Also, the restricted Weyl group W_e is the group $S_{m_1} \times S_{m_2} \times \dots$, where m_i denotes the number of parts of λ that equal i, acting on E_e by permuting all the ε_i of equal length.

As explained in [**6**], there is a convenient way to visualize the good grading $\Gamma(p)$ corresponding to $p = (p_1, \dots, p_m) \in \mathscr{P}_e$. First, associate a pyramid $\pi(p)$ to p by sliding all numbered boxes in row r_i of the Dynkin pyramid to the right by p_i units, for each $i = 1, \dots, m$; for example, $\pi(0)$ is the Dynkin pyramid itself. Then, $\Gamma(p)$ is the grading induced by declaring that each matrix unit $e_{i,j}$ is of degree $\mathrm{col}(i) - \mathrm{col}(j)$, the notation $\mathrm{col}(i)$ now denoting the column number of the ith box in $\pi(p)$. Rearranging the numbers in the boxes of $\pi(p)$ so that $\mathrm{col}(1) \geqslant \mathrm{col}(2) \geqslant \dots \geqslant \mathrm{col}(n)$, we find that the characteristic of $\Gamma(p)$ is $(c_1, \dots, c_{n-1})$ where $c_i = \mathrm{col}(i) - \mathrm{col}(i+1)$. Finally, recall that $p \in \mathscr{P}_e$ defines an integral good grading if and only if p lies on as many affine hyperplanes as the origin. For every $p \in \mathscr{P}_e$, there is a point $p' \in \mathscr{P}_e$ lying in the closure of the alcove containing p, such that $\Gamma(p')$ is an integral good grading. This means that in type A, one can always restrict attention just to integral good gradings without losing any generality. Moreover, every integral good grading is adjacent to an even good grading, as was noted already in the introduction of [**3**]. We note that these nice things definitely do not usually happen in other types, as can be observed for the symplectic and orthogonal groups in the next two sections.

EXAMPLE 28. Let $\lambda = (3, 3, 2)$ as above. The set $\mathscr{P}_e$ consists of all $p = (p_1, p_2, p_3)$ with $3p_1 + 3p_2 + 2p_3 = 0$, $|p_1 - p_2| < 1$, $|p_2 - p_3| < 2$ and $|p_1 - p_3| < 2$:



The Weyl group $W_e \cong S_2$ is generated by the reflection in the horizontal axis in this picture. The alcoves in $\mathscr{P}_e$ are the interiors of the fourteen triangles. There are just three points that lie on as many affine hyperplanes as the origin, with associated pyramids (renumbered so that we can read off their characteristics):



Hence, there are three conjugacy classes of integral good gradings for e, with characteristics $(0, 2, 0, 0, 2, 0, 0)$, $(0, 1, 1, 0, 1, 1, 0)$ and $(0, 0, 2, 0, 0, 2, 0)$, respectively.

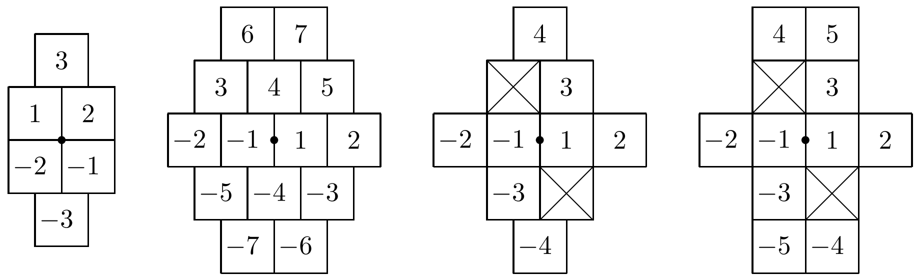
7. *Good gradings for* $\mathfrak{sp}_{2n}(\mathbb{C})$

Next, we discuss $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$. Let V denote the natural $2n$-dimensional $\mathfrak{g}$-module with standard basis $v_1, \dots, v_n, v_{-n}, \dots, v_{-1}$ and $\mathfrak{g}$-invariant skew-symmetric bilinear form $(\,,)$ defined by $(v_i, v_j) = (v_{-i}, v_{-j}) = 0$ and $(v_i, v_{-j}) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant n$. Let $\mathfrak{t}$ be the set of all elements of $\mathfrak{g}$ which act diagonally on the standard basis of V. For the simple roots $\Delta \subset \Phi$, we take $\alpha_1 = \delta_1 - \delta_2, \dots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = 2\delta_n$, where $\delta_i \in \mathfrak{t}^*$ is defined from $hv_i = \delta_i(h)v_i$ for each $h \in \mathfrak{t}$ and $i = 1, \dots, n$. Writing $e_{i,j}$ for the ij-matrix unit, we see that the following matrices give a Chevalley basis for $\mathfrak{g}$:

$$\{e_{i,j} - e_{-j,-i}\}_{1 \leqslant i,j \leqslant n} \cup \{e_{i,-j} + e_{j,-i}, e_{-i,j} + e_{-j,i}\}_{1 \leqslant i < j \leqslant n} \cup \{e_{k,-k}, e_{-k,k}\}_{1 \leqslant k \leqslant n}.$$

Let $\sigma_{i,j} \in \{\pm 1\}$ denote the $e_{i,j}$-coefficient of the unique element in the above basis that involves $e_{i,j}$.

Nilpotent orbits in $\mathfrak{g}$ are parameterized by partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots)$ of $2n$ such that every odd part appears with even multiplicity. Fix such a symplectic partition λ throughout the section. We begin by introducing the *symplectic Dynkin pyramid* of shape λ, following the idea of [**6**] closely once more. This is a diagram consisting of $2n$ boxes each of size 2 units by 2 units drawn in the xy-plane. As before, the coordinates of a box are the coordinates of its midpoint, and the row and column numbers of a box mean its y- and x-coordinate, respectively. Before we attempt a formal definition, here are some examples of symplectic Dynkin pyramids for $\lambda = (2,2,1,1), (4,3,3,2,2), (4,2,1,1)$ and $(4,2,2,2)$, respectively:



To describe the Dynkin pyramid in the general case, the parts of λ indicate the number of boxes in each row, and the rows are added to the diagram in order, starting with the row corresponding to the largest part of λ closest to the x-axis and moving out from there, in a centrally symmetric way. The only complication is that if some (necessarily even) part λ_i of λ has odd multiplicity, then the first time a row of this length is added to the diagram it is split into two halves, the right half is added to the next free row in the upper half plane in columns $1, 3, \ldots, \lambda_i - 1$ and the left half is added to the lower half plane in a centrally symmetric way. We refer to the exceptional rows arising in this way as *skew rows*; in particular, if the largest part of λ has odd multiplicity, then the zeroth row is a skew row. The missing boxes in skew rows are drawn as a box with a cross through it. We let $r_1 < \ldots < r_m$ denote the numbers of the non-empty rows in the upper half plane that are not skew rows, and define $\bar{\lambda}_i$ to be the number of boxes in row r_i for each $i = 1, \ldots, m$.

Fix from now on a numbering of the boxes of the Dynkin pyramid by the numbers $1, \ldots, n, -n, \ldots, -1$ in such a way that i and $-i$ appear in centrally symmetric boxes, for each $i = 1, \ldots, n$. As before, we write $\mathrm{row}(i)$ and $\mathrm{col}(i)$ for the row and column numbers of the ith box. Now we can fix a choice of an $\mathfrak{sl}_2$-triple (e, h, f) with e of Jordan type λ. Define $e \in \mathfrak{g}$ to be the matrix $\sum_{i,j} \sigma_{i,j} e_{i,j}$, where the sum is over all pairs i, j of boxes in the Dynkin pyramid such that

$$\begin{aligned} &\textit{either} \quad \mathrm{col}(i) = \mathrm{col}(j) + 2 \quad \text{and} \quad \mathrm{row}(i) = \mathrm{row}(j); \\ &\textit{or} \quad \mathrm{col}(i) = 1, \quad \mathrm{col}(j) = -1 \quad \text{and} \quad \mathrm{row}(i) = -\mathrm{row}(j) \text{ is a skew} \\ &\qquad \text{row in the upper half plane.} \end{aligned}$$

For example, if $\lambda = (4,2,1,1)$ and the Dynkin pyramid is labelled as above, then

$$e = e_{3,-3} + e_{2,1} + e_{1,-1} - e_{-1,-2}.$$

Also define $h \in \mathfrak{t}$ to be $\sum_i \mathrm{col}(i) e_{i,i}$, again summing over all boxes in the Dynkin pyramid. There is then a unique $f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

The important thing about these choices is that once again $\mathfrak{t}$ is contained in $\mathfrak{c}$ and $\mathfrak{t}_e$ is a Cartan subalgebra of $\mathfrak{c}_e$. In fact, E_e consists of all matrices in $\mathfrak{t}$ with entries from $\mathbb{R}$, such that the ith and jth diagonal entries are equal whenever $\mathrm{row}(i) = \mathrm{row}(j)$ and the kth diagonal entry is zero whenever $\mathrm{row}(k)$ is a skew row, for $1 \leqslant i, j, k \leqslant n$. For $i = 1, \ldots, m$, let ε_i be the function picking out the jth diagonal entry of a matrix in E_e, where $1 \leqslant j \leqslant n$ here is chosen so that the jth box is in row r_i in the Dynkin pyramid. Then, $\varepsilon_1, \ldots, \varepsilon_m$ form a basis for E_e^*. We identify E_e and E_e^* via the trace form $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}/2\bar{\lambda}_i$. We have

$$\Phi_e = \{\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leqslant i, j, k \leqslant m, i \neq j\}$$

if there are no skew rows, that is, all non-zero parts of λ are of even multiplicity, or

$$\Phi_e = \{\pm \varepsilon_h, \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leqslant h, i, j, k \leqslant m, i \neq j\}$$

if there are skew rows. The values of $d(\alpha)$ from Theorem 20 for all $\alpha \in \Phi_e$ are:

$$d(\varepsilon_i \pm \varepsilon_j) = 1 + |\bar{\lambda}_i - \bar{\lambda}_j|,$$

$$d(\pm 2\varepsilon_k) = \begin{cases} 1 & \text{if } \bar{\lambda}_k \text{ is odd,} \\ 3 & \text{if } \bar{\lambda}_k \text{ is even,} \end{cases}$$

$$d(\pm\varepsilon_h) = 1 + \min\{|\bar{\lambda}_h - t| \mid t \text{ is a non-zero part of } \lambda \text{ of odd multiplicity}\}.$$

Hence, representing a point $p \in E_e$ as an m-tuple $p = (p_1, \dots, p_m)$ of real numbers defined from $p_i = \varepsilon_i(p)$, we see that the good grading polytope $\mathscr{P}_e$ is the open subset of E_e defined by the inequalities

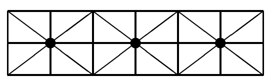
$$|p_i \pm p_j| < 1 + \bar{\lambda}_i - \bar{\lambda}_j,$$

$$|p_k| < \begin{cases} \frac{1}{2} & \text{if } \bar{\lambda}_k \text{ is odd,} \\ 1 & \text{if } \bar{\lambda}_k \text{ is even of multiplicity greater than 2 in } \lambda, \\ \frac{3}{2} & \text{if } \bar{\lambda}_k \text{ is even of multiplicity 2 in } \lambda, \end{cases}$$

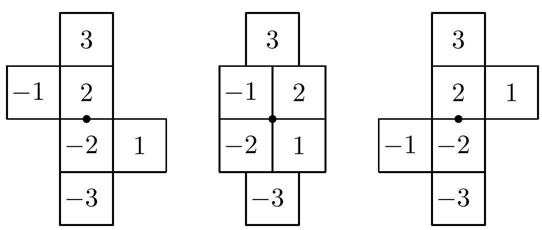
for all $1 \leqslant i < j \leqslant m$ and $1 \leqslant k \leqslant m$. Also, letting $\overline{m}_i$ denote the multiplicity of i as a part of the partition $(\bar{\lambda}_1, \bar{\lambda}_2, \dots)$, we deduce that the restricted Weyl group W_e is the subgroup $B_{\overline{m}_1} \times B_{\overline{m}_2} \times \dots$ of the Weyl group B_m, acting on $\{\pm\varepsilon_1, \dots, \pm\varepsilon_m\}$ by all sign changes and all permutations of the ε_i of equal length.

Again, there is a useful combinatorial way to visualize the good grading $\Gamma(p)$ corresponding to a point $p = (p_1, \dots, p_m) \in \mathscr{P}_e$ involving pyramids. Let $\pi(p)$ denote the pyramid obtained from the symplectic Dynkin pyramid by sliding all numbered boxes in rows $\pm r_i$ to the right by $\pm p_i$ units. Then, $\Gamma(p)$ is the grading induced by declaring that each matrix unit $e_{i,j}$ is of degree $\mathrm{col}(i) - \mathrm{col}(j)$, where the notation $\mathrm{col}(i)$ now denotes the column number of the ith box in $\pi(p)$. The characteristic of the grading $\Gamma(p)$ can be computed by first rearranging the entries in the boxes of $\pi(p)$ using all permutations and sign changes from the Weyl group $W = B_n$, so that $\mathrm{col}(1) \geqslant \mathrm{col}(2) \geqslant \dots \geqslant \mathrm{col}(n) \geqslant 0$. Then, the characteristic of $\Gamma(p)$ is $(c_1, \dots, c_n)$ where $c_i = \mathrm{col}(i) - \mathrm{col}(i+1)$ for $i = 1, \dots, n-1$ and $c_n = 2\,\mathrm{col}(n)$.

Example 29. Take $\lambda = (2, 2, 1, 1)$. Then $\mathscr{P}_e$ consists of all $p = (p_1, p_2)$ with $|p_1| < \frac{3}{2}$ and $|p_2| < \frac{1}{2}$:



The Weyl group $W_e \cong S_2 \times S_2$ is generated by reflections in the horizontal and vertical axes. There are three integral good gradings for e compatible with $\mathfrak{t}$, with associated pyramids (renumbered so that we can read off their characteristics):



These have characteristics $(2, 0, 0)$, $(0, 1, 0)$ and $(2, 0, 0)$, respectively. The right and left good gradings are conjugate by the element of W_e corresponding to the reflection in the vertical axes of the good grading polytope.

8. *Good gradings for* $\mathfrak{so}_N(\mathbb{C})$

Finally, let $\mathfrak{g} = \mathfrak{so}_N(\mathbb{C})$ and set $n = \lfloor \frac{1}{2}N \rfloor$, assuming $N \geqslant 3$. Let V be the natural N-dimensional $\mathfrak{g}$-module with standard basis $v_1, \dots, v_n, v_0, v_{-n}, \dots, v_{-1}$ and $\mathfrak{g}$-invariant symmetric bilinear form $(\,,)$ defined by

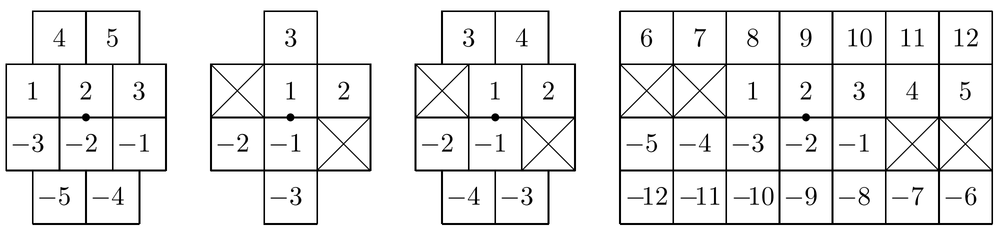
$$(v_0, v_i) = (v_0, v_{-i}) = 0, \quad (v_0, v_0) = 2, \quad (v_i, v_j) = (v_{-i}, v_{-j}) = 0 \quad \text{and} \quad (v_i, v_{-j}) = \delta_{i,j}$$

for $1 \leqslant i, j \leqslant n$ (omitting v_0 everywhere if N is even). Let $\mathfrak{t}$ be the set of all elements of $\mathfrak{g}$ which act diagonally on the standard basis of V. Defining $\delta_i \in \mathfrak{t}^*$ as in Section 7, we find that a choice of simple roots $\Delta \subset \Phi$ is given by $\alpha_1 = \delta_1 - \delta_2$, …, $\alpha_{n-1} = \delta_{n-1} - \delta_n$, and $\alpha_n = \delta_{n-1} + \delta_n$ if N is even or $\alpha_n = \delta_n$ if N is odd. The following matrices give a Chevalley basis for $\mathfrak{g}$ (again omitting the last family if N is even):

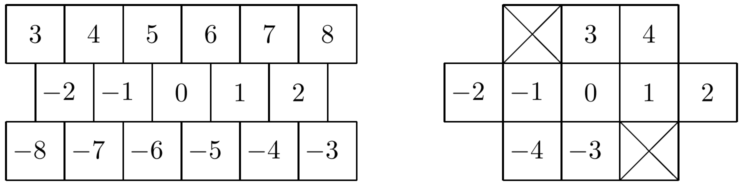
$$\begin{aligned}\{e_{i,j} - e_{-j,-i}\}_{1 \leqslant i,j \leqslant n} &\cup \{e_{i,-j} - e_{j,-i}, e_{-j,i} - e_{-i,j}\}_{1 \leqslant i < j \leqslant n} \\ &\cup \{2e_{k,0} - e_{0,-k}, e_{0,k} - 2e_{-k,0}\}_{1 \leqslant k \leqslant n}.\end{aligned}$$

As before, define $\sigma_{i,j}$ to be the coefficient of $e_{i,j}$ in this basis if it appears, or zero if no basis element involves $e_{i,j}$.

Nilpotent orbits in $\mathfrak{g}$ are parameterized by partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots)$ of N such that every even part appears with even multiplicity; in case N is even, we mean nilpotent orbits under the group O_N not SO_N here. Fix such an orthogonal partition λ throughout the section. We need the *orthogonal Dynkin pyramid* of type λ, which again consists of N boxes of size 2 units by 2 units arranged in the xy-plane in a centrally symmetric way. Assume to start with that N is even. Then the Dynkin pyramid is constructed as in the symplectic case, adding rows of lengths determined by the parts of λ working outwards from the x-axis starting with the largest part, in a centrally symmetric way. The only difficulty is if some (necessarily odd) part of λ appears with odd multiplicity. As N is even, the number of distinct parts having odd multiplicity is even. Choose $i_1 < j_1 < \dots < i_r < j_r$ such that $\lambda_{i_1} > \lambda_{j_1} > \dots > \lambda_{i_r} > \lambda_{j_r}$ are representatives for all the distinct odd parts of λ having odd multiplicity. Then the first time the part λ_{i_s} needs to be added to the diagram, the part λ_{j_s} is also added at the same time, so that the parts λ_{i_s} and λ_{j_s} of λ contribute two centrally symmetric rows to the diagram, one row in the upper half plane with boxes in columns $1 - \lambda_{j_s}, 3 - \lambda_{j_s}, \dots, \lambda_{i_s} - 1$ and the other row in the lower half plane with boxes in columns $1 - \lambda_{i_s}, 3 - \lambda_{i_s}, \dots, \lambda_{j_s} - 1$. We will refer to the exceptional rows arising in this way as *skew rows*. To demonstrate this construction we give examples, for $\lambda = (3, 3, 2, 2), (3, 1, 1, 1), (3, 2, 2, 1)$ and $(7, 7, 7, 3)$, respectively:



If N is odd, there is one additional consideration. There must be some odd part appearing with odd multiplicity. Let λ_i be the largest such part, and put λ_i boxes into the zeroth row in columns $1 - \lambda_i, 3 - \lambda_i, \dots, \lambda_i - 1$; we also treat this zeroth row as a skew row. Now remove the part λ_i from λ, to obtain a partition of an even number. The remaining parts are then added to the diagram exactly as in the case N even. Below we give two more examples of orthogonal pyramids, for $\lambda = (6, 6, 5)$ and $(5, 3, 1)$, respectively:



Let $r_1 < \ldots < r_m$ denote the numbers of the non-empty rows in the upper half plane of the Dynkin pyramid that are not skew rows, and define $\bar{\lambda}_i$ to be the number of boxes in row r_i for each $i = 1, \ldots, m$.

Note that in the case N is odd, there is always a box at $(0, 0)$; we always number it by 0. The remaining boxes, for N even or odd, are numbered $\pm 1, \ldots, \pm n$ exactly as in the symplectic case, and we use the notation $\mathrm{row}(i)$ and $\mathrm{col}(i)$ just as before. Define $e \in \mathfrak{g}$ to be the matrix $\sum_{i,j} \sigma_{i,j} e_{i,j}$, where the sum is over all pairs i, j of boxes in the Dynkin pyramid such that

$$
\begin{aligned}
\textit{either}\quad & \mathrm{col}(i) = \mathrm{col}(j) + 2 \quad \text{and} \quad \mathrm{row}(i) = \mathrm{row}(j);\\
\textit{or}\quad & \mathrm{col}(i) = 2, \quad \mathrm{col}(j) = 0 \quad \text{and} \quad \mathrm{row}(i) = -\,\mathrm{row}(j) \text{ is a skew}\\
& \text{row in the upper half plane;}\\
\textit{or}\quad & \mathrm{col}(i) = 0, \quad \mathrm{col}(j) = -2 \quad \text{and} \quad \mathrm{row}(i) = -\,\mathrm{row}(j) \text{ is a skew}\\
& \text{row in the upper half plane.}
\end{aligned}
$$

This is an element of $\mathfrak{g}$ having Jordan type λ. (If *all* parts of λ are even then there is another conjugacy class of elements of $\mathfrak{g}$ of Jordan type λ, a representative for which can be obtained using the above formula by swapping the entries i and $-i$ in the Dynkin pyramid for some $1 \leqslant i \leqslant n$.) For example, for $\lambda = (5, 3, 1)$ labelled as above,

$$e = e_{4,3} - e_{4,-3} + e_{3,-4} - e_{-3,-4} + e_{2,1} + e_{1,0} - e_{0,-1} - e_{-1,-2}.$$

Let $h = \sum_i \mathrm{col}(i) e_{i,i}$. Then there is a unique element $f \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple.

Again, these choices ensure that $\mathfrak{t}$ is contained in $\mathfrak{c}$ and $\mathfrak{t}_e$ is a Cartan subalgebra of $\mathfrak{c}_e$. In the same way as for $\mathfrak{sp}_{2n}(\mathbb{C})$, E_e consists of all matrices in $\mathfrak{t}$ with entries from $\mathbb{R}$, such that the ith and jth diagonal entries are equal whenever $\mathrm{row}(i) = \mathrm{row}(j)$ and the kth diagonal entry is zero whenever $\mathrm{row}(k)$ is a skew row, for $1 \leqslant i, j, k \leqslant n$. Define the basis $\varepsilon_1, \ldots, \varepsilon_m$ for E_e^* just as in the symplectic case, and work with the inner product defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}/2\bar{\lambda}_i$. This time, we have

$$\Phi_e = \{\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leqslant i, j, k \leqslant m, i \neq j, \bar{\lambda}_k \neq 1\}$$

if there are no skew rows, that is, all non-zero parts of λ are of even multiplicity, or

$$\Phi_e = \{\pm\varepsilon_h, \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leqslant h, i, j, k \leqslant m, i \neq j, \bar{\lambda}_k \neq 1\}$$

if there are skew rows. The values of $d(\alpha)$ from Theorem 20 for all $\alpha \in \Phi_e$ are:

$$
\begin{aligned}
d(\varepsilon_i \pm \varepsilon_j) &= 1 + |\bar{\lambda}_i - \bar{\lambda}_j|,\\
d(\pm 2\varepsilon_k) &= \begin{cases} 1 & \text{if } \bar{\lambda}_k \text{ is even,}\\ 3 & \text{if } \bar{\lambda}_k \text{ is odd,} \end{cases}\\
d(\pm\varepsilon_h) &= 1 + \min\{|\bar{\lambda}_h - t| \mid t \text{ is a non-zero part of } \lambda \text{ of odd multiplicity}\}.
\end{aligned}
$$

Hence, representing points $p \in E_e$ as m-tuples $p = (p_1, \ldots, p_m)$ of real numbers so that $p_i = \varepsilon_i(p)$, the good grading polytope $\mathscr{P}_e$ is the open subset of E_e defined by the inequalities

$$|p_i \pm p_j| < 1 + \bar{\lambda}_i - \bar{\lambda}_j,$$

$$|p_k| < \begin{cases} \frac{1}{2} & \text{if } \bar{\lambda}_k \text{ is even,} \\ 1 & \text{if } \bar{\lambda}_k \text{ is odd of multiplicity greater than 2 in } \lambda, \\ \frac{3}{2} & \text{if } \bar{\lambda}_k \neq 1 \text{ is odd of multiplicity 2 in } \lambda, \\ s & \text{if } \bar{\lambda}_k = 1 \text{ is of multiplicity 2 in } \lambda, \end{cases}$$

for all $1 \leqslant i < j \leqslant m$ and $1 \leqslant k \leqslant m$. Here, in the case that the part 1 is of multiplicity 2 in λ, s denotes the smallest part of λ that is greater than 1. Let $\overline{m}_i$ denote the multiplicity of i as a part of the partition $(\bar{\lambda}_1, \bar{\lambda}_2, \ldots)$. If there are skew rows, then the restricted Weyl group W_e is $B_{\overline{m}_1} \times B_{\overline{m}_2} \times \ldots$, acting on $\{\pm\varepsilon_1, \ldots, \pm\varepsilon_m\}$ by all sign changes and all permutations of the ε_i of equal length. If there are no skew rows, then W_e is instead the subgroup of $B_{\overline{m}_1} \times B_{\overline{m}_2} \times \ldots$ consisting of all the elements in this group that act on $\{\pm\varepsilon_1, \ldots, \pm\varepsilon_m\}$ with only an even number of sign changes of the ε_i for which $\bar{\lambda}_i$ is odd.

For $p = (p_1, \ldots, p_m) \in \mathscr{P}_e$, we define the pyramid $\pi(p)$ by sliding all numbered boxes in rows $\pm r_i$ of the orthogonal Dynkin pyramid to the right by $\pm p_i$ units. Then, the grading $\Gamma(p)$ associated to the point $p \in \mathscr{P}_e$ is the grading induced by declaring that each matrix unit $e_{i,j}$ is of degree $\mathrm{col}(i) - \mathrm{col}(j)$, where the notation $\mathrm{col}(i)$ here denotes the column number of the ith box in $\pi(p)$. To compute the characteristic of the grading $\Gamma(p)$, suppose first that N is odd. Rearrange the entries in the boxes of $\pi(p)$ using all permutations and sign changes from the Weyl group $W = B_n$ so that $\mathrm{col}(1) \geqslant \mathrm{col}(2) \geqslant \ldots \geqslant \mathrm{col}(n) \geqslant 0$. Then, the characteristic of $\Gamma(p)$ is $(c_1, \ldots, c_n)$ where $c_i = \mathrm{col}(i) - \mathrm{col}(i+1)$ for $i = 1, \ldots, n-1$ and $c_n = \mathrm{col}(n)$. Instead, if N is even, rearrange the entries in the boxes of $\pi(p)$ using all permutations and sign changes from the Weyl group $W = D_n$ (that is, so that there are only an even number of sign changes in total) so that $\mathrm{col}(1) \geqslant \ldots \geqslant \mathrm{col}(n-1) \geqslant |\mathrm{col}(n)|$. Then, the characteristic of $\Gamma(p)$ is $(c_1, \ldots, c_n)$ where $c_i = \mathrm{col}(i) - \mathrm{col}(i+1)$ for $i = 1, \ldots, n-1$ and $c_n = \mathrm{col}(n-1) + \mathrm{col}(n)$.

Acknowledgements. We thank Ross Lawther, Gerhard Röhrle, Gary Seitz, Eric Sommers and Sergey Yuzvinsky for help.

References

1. K. Baur and N. Wallach, 'Nice parabolic subalgebras of reductive Lie algebras', *Represent. Theory* 9 (2005) 1–29.

2. J. de Boer and T. Tjin, 'Quantization and representation theory of finite W-algebras', *Comm. Math. Phys.* 158 (1993) 485–516.

3. J. Brundan and A. Kleshchev, 'Shifted Yangians and finite W-algebras', *Adv. Math.* 200 (2006) 136–195.

4. R. W. Carter, *Finite groups of Lie type* (Wiley, New York, 1993).

5. J. M. Douglass, 'The adjoint representation of a reductive group and hyperplane arrangements', *Represent. Theory* 3 (1999) 444–456.

6. A. G. Elashvili and V. G. Kac, 'Classification of good gradings of simple Lie algebras', *Lie groups and invariant theory* (ed. E. B. Vinberg), American Mathematical Society Translations (2) 213 (AMS, Providence, RI, 2005) 85–104.

7. W. L. Gan and V. Ginzburg, 'Quantization of Slodowy slices', *Internat. Math. Res. Notices* 5 (2002) 243–255.

8. The GAP Group, 'GAP – Groups, Algorithms, and Programming', http://www.gap-system.org, Version 4.3, 2002.

9. R. Howlett, 'Normalizers of parabolic subgroups of reflection groups', *J. London Math. Soc.* (2) 21 (1980) 62–80.

10. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics 9 (Springer, New York, 1994).

11. J. C. Jantzen, 'Nilpotent orbits in representation theory', *Lie theory*, Progress in Mathematics 228 (Birkhäuser, Boston, 2004) 1–211.
12. V. Kac, S. Roan and M. Wakimoto, 'Quantum reduction for affine superalgebras', *Comm. Math. Phys.* 241 (2003) 307–342.
13. B. Kostant, 'On Whittaker modules and representation theory', *Invent. Math.* 48 (1978) 101–184.
14. T. E. Lynch, 'Generalized Whittaker vectors and representation theory', PhD Thesis, M.I.T., 1979.
15. P. Orlik and L. Solomon, 'Coxeter arrangements', *Singularities* (ed. P. Orlik), Proceedings of Symposia in Pure Mathematics 40 (American Mathematical Society, Providence, RI, 1983) 269–292.
16. P. Orlik and H. Terao, *Arrangements of hyperplanes* (Springer, New York, 1992).
17. P. Orlik and H. Terao, 'Coxeter arrangements are hereditarily free', *Tôhoku Math. J.* 45 (1993) 369–383.
18. A. Premet, 'Special transverse slices and their enveloping algebras', *Adv. Math.* 170 (2002) 1–55.
19. A. Premet, 'Enveloping algebras of Slodowy slices and the Joseph ideal', Preprint, 2005, arXiv:math.RT/0504343.
20. E. Sommers, 'A family of affine Weyl group representations', *Transform. Groups* 2 (1997) 375–390.

Jonathan Brundan
Department of Mathematics
University of Oregon
Eugene, OR 97403
USA

brundan@darkwing.uoregon.edu

Simon M. Goodwin
Institut for Matematiske Fag
Aarhus Universitet
DK-8000 Aarhus C
Denmark

Current address:
School of Mathematics
University of Birmingham
Birmingham
B15 2TT
United Kingdom

goodwin@maths.bham.ac.uk