THE DEGENERATE HEISENBERG CATEGORY AND ITS GROTHENDIECK RING

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Abstract. The degenerate Heisenberg category $\mathcal{H}eis_k$ is a strict monoidal category which was originally introduced in the special case $k = -1$ by Khovanov in 2010. Khovanov conjectured that the Grothendieck ring of the additive Karoubi envelope of his category is isomorphic to a certain $\mathbb{Z}$-form for the universal enveloping algebra of the infinite-dimensional Heisenberg Lie algebra specialized at central charge -1. We prove this conjecture and extend it to arbitrary central charge $k \in \mathbb{Z}$. We also explain how to categorify the comultiplication (generically).

1. Introduction

Throughout the article, we work over a fixed ground field $\Bbbk$ of characteristic zero. The degenerate Heisenberg category $\mathcal{H}eis_k$ of central charge $k \in \mathbb{Z}$ is a strict $\Bbbk$-linear monoidal category which was introduced originally by Khovanov [Kh] in the special case $k = -1$, motivated by the calculus of induction and restriction functors between representations of the symmetric groups. Khovanov's definition of $\mathcal{H}eis_k$ was extended to arbitrary central charge in [MS, B]. The relations of this category are modeled on those of a $\mathbb{Z}$-form Heis_k for a central reduction of the universal enveloping algebra $U(\mathfrak{h})$ of the infinite-dimensional Heisenberg Lie algebra. By [Kh, MS], there is an injective ring homomorphism

$$\gamma_k : \mathrm{Heis}_k \to K_0(\mathrm{Kar}(\mathcal{H}eis_k)) \tag{1.1}$$

to the Grothendieck ring of the additive Karoubi envelope of $\mathcal{H}eis_k$. In this paper, we prove that γ_k is also surjective, so that $\mathcal{H}eis_k$ *categorifies* Heis_k, as was conjectured in [Kh, MS]. We also take a first step towards categorification of the comultiplication on $U(\mathfrak{h})$.

To give more precise statements, we need to recall some basic notions. Let $\mathrm{Sym}_\mathbb{Z}$ be the ring of *symmetric functions*; see [M]. It is freely generated either by the *elementary symmetric functions* $\{e_n\}_{n\geq 1}$ or the *complete symmetric funtions* $\{h_n\}_{n\geq 1}$. We also have the *power sums* $\{p_n\}_{n\geq 1}$ whose images generate $\mathrm{Sym}_\mathbb{Q} := \mathbb{Q} \otimes_\mathbb{Z} \mathrm{Sym}_\mathbb{Z}$. Moreover, $\mathrm{Sym}_\mathbb{Z}$ is a Hopf ring with comultiplication $\delta : \mathrm{Sym}_\mathbb{Z} \to \mathrm{Sym}_\mathbb{Z} \otimes_\mathbb{Z} \mathrm{Sym}_\mathbb{Z}, f \mapsto \sum_{(f)} f_{(1)} \otimes f_{(2)}$ satisfying

$$\delta(h_n) = \sum_{r=0}^{n} h_{n-r} \otimes h_r, \qquad \delta(e_n) = \sum_{r=0}^{n} e_{n-r} \otimes e_r, \qquad \delta(p_n) = p_n \otimes 1 + 1 \otimes p_n, \tag{1.2}$$

where $h_0 = e_0 = 1$ by convention. As a $\mathbb{Z}$-module, $\mathrm{Sym}_\mathbb{Z}$ is free with the canonical basis $\{s_\lambda\}_{\lambda\in\mathcal{P}}$ of *Schur functions* indexed by the set $\mathcal{P}$ of all partitions.

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The *infinite-dimensional Heisenberg Lie algebra* $\mathfrak{h}$ is the Lie algebra over $\mathbb{Q}$ generated by $\left\{c, p_n^\pm \,\middle|\, n \geq 1\right\}$ subject to the relations

$$[c, p_n^\pm] = [p_m^+, p_n^+] = [p_m^-, p_n^-] = 0, \qquad\qquad [p_m^+, p_n^-] = \delta_{m,n} nc. \tag{1.3}$$

The central reduction $U(\mathfrak{h})/(c - k)$ of its universal enveloping algebra may also be realized as the *Heisenberg double* $\mathrm{Sym}_\mathbb{Q} \#_\mathbb{Q} \mathrm{Sym}_\mathbb{Q}$ with respect to the bilinear Hopf pairing

$$\langle -, - \rangle_k \colon \mathrm{Sym}_\mathbb{Q} \times \mathrm{Sym}_\mathbb{Q} \to \mathbb{Q}, \quad \langle p_m, p_n \rangle_k = \delta_{m,n} nk. \tag{1.4}$$

By definition, $\mathrm{Sym}_\mathbb{Q} \#_\mathbb{Q} \mathrm{Sym}_\mathbb{Q}$ is the vector space $\mathrm{Sym}_\mathbb{Q} \otimes_\mathbb{Q} \mathrm{Sym}_\mathbb{Q}$ with associative multiplication defined by

$$(e \otimes f)(g \otimes h) := \sum_{(f),(g)} \langle f_{(1)}, g_{(2)} \rangle_k \, e g_{(1)} \otimes f_{(2)} h.$$

The pairing of two complete symmetric functions is an integer, as follows for example by comparing the coefficients appearing in [S, Th. 5.3] to [S, (2.2)]. Thus we can restrict to obtain a biadditive form $\langle -, - \rangle_k : \mathrm{Sym}_\mathbb{Z} \times \mathrm{Sym}_\mathbb{Z} \to \mathbb{Z}$. The resulting Heisenberg double

$$\mathrm{Heis}_k := \mathrm{Sym}_\mathbb{Z} \#_\mathbb{Z} \mathrm{Sym}_\mathbb{Z} \tag{1.5}$$

gives us a natural $\mathbb{Z}$-form for $U(\mathfrak{h})/(c - k) \cong \mathrm{Sym}_\mathbb{Q} \#_\mathbb{Q} \mathrm{Sym}_\mathbb{Q}$. For $f \in \mathrm{Sym}_\mathbb{Z}$, we write f^- and f^+ for the elements $f \otimes 1$ and $1 \otimes f$ of Heis_k, respectively. Then Heis_k is generated as a ring by the elements $\{h_n^+, e_n^-\}_{n \geq 0}$ subject to the relations

$$h_0^+ = e_0^- = 1, \qquad h_m^+ h_n^+ = h_n^+ h_m^+, \qquad e_m^- e_n^- = e_n^- e_m^-, \qquad h_m^+ e_n^- = \sum_{r=0}^{\min(m,n)} \binom{k}{r} e_{n-r}^- h_{m-r}^+. \tag{1.6}$$

See [S, Section 5] and [LRS, Appendix A] where this and other presentations are derived. The usual comultiplication on $U(\mathfrak{h})$ descends to ring homomorphisms

$$\delta_{l|m} : \mathrm{Heis}_k \to \mathrm{Heis}_l \otimes_\mathbb{Z} \mathrm{Heis}_m, \quad f^\pm \mapsto \sum_{(f)} (f_{(1)})^\pm \otimes (f_{(2)})^\pm \tag{1.7}$$

for $k = l + m$ and $f \in \mathrm{Sym}_\mathbb{Z}$. The antipode induces $\sigma_k : \mathrm{Heis}_k \xrightarrow{\sim} (\mathrm{Heis}_{-k})^{\mathrm{op}}$, $s_\lambda^\pm \mapsto (-1)^{|\lambda|} s_{\lambda^T}^\pm$. Also there is an isomorphism $\omega_k : \mathrm{Heis}_k \xrightarrow{\sim} \mathrm{Heis}_{-k}$, $s_\lambda^\pm \mapsto s_{\lambda^T}^\mp$.

The *degenerate Heisenberg category* $\mathcal{H}eis_k$ is a strict $\mathbb{k}$-linear monoidal category with two generating objects $\uparrow$ and $\downarrow$ and six generating morphisms

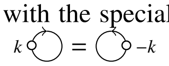
A full set of relations between these generating morphisms is recorded in Definition 5.1 below, where we adopt the usual string calculus for strict monoidal categories. The relations imply that $\mathcal{H}eis_k$ is *strictly pivotal* with duality functor $*$ defined on a morphism by rotating its string diagram through 180°. In particular, the generating objects $\uparrow$ and $\downarrow$ are duals of each other. Letting $\mathfrak{S}_n$ denote the symmetric group with basic transpositions $s_1, \dots, s_{n-1}$, there is also an algebra homomorphism $\imath_n : \mathbb{k}\mathfrak{S}_n \to \mathrm{End}_{\mathcal{H}eis_k}(\uparrow^{\otimes n})$, which sends s_i to the crossing of the ith and $(i+1)$th strings. Note we always number strings in diagrams by $1, 2, \dots$ from *right to left*.

By the *additive Karoubi envelope* $\mathrm{Kar}(\mathcal{H}eis_k)$ of $\mathcal{H}eis_k$, we mean the idempotent completion of its additive envelope $\mathrm{Add}(\mathcal{H}eis_k)$. Let $K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ be the Grothendieck ring of the monoidal category $\mathrm{Kar}(\mathcal{H}eis_k)$, i.e., the split Grothendieck group with multiplication $[X][Y] := [X \otimes Y]$. For $\lambda \in \mathcal{P}$ with $|\lambda| = n$, let $e_\lambda \in \mathbb{k}\mathfrak{S}_n$ be the corresponding Young symmetrizer, so that the left ideal $S(\lambda) := (\mathbb{k}\mathfrak{S}_n)e_\lambda$ is the usual (irreducible) *Specht module* for the symmetric group. Associated to the idempotent e_λ, we also have the object

$$S_\lambda^+ := \left(\uparrow^{\otimes n}, \imath_n(e_\lambda)\right) \in \mathrm{Kar}(\mathcal{H}eis_k). \tag{1.8}$$

Let $S_\lambda^- := (S_\lambda^+)^*$, and set $H_n^\pm := S_{(n)}^\pm$ and $E_n^\pm := S_{(1^n)}^\pm$ for short. Our first main result is as follows.

Theorem 1.1. *There is a ring isomorphism* $\gamma_k : \mathrm{Heis}_k \xrightarrow{\sim} K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ *such that* $s^\pm_\lambda \mapsto [S^\pm_\lambda]$ *for each* $\lambda \in \mathcal{P}$. *In particular,* $h^\pm_n \mapsto [H^\pm_n]$ *and* $e^\pm_n \mapsto [E^\pm_n]$. *Also for* $X \in \mathrm{Kar}(\mathcal{H}eis_k)$ *we have that* $[X] = 0 \Rightarrow X = 0$.

This proves extended versions of [Kh, Conjecture 1] and [MS, Conjecture 4.5]. The original conjectures in *loc. cit.* are concerned with the specialization $\mathcal{H}eis_k(\delta)$ of $\mathcal{H}eis_k$ obtained by evaluating the (strictly central) bubble  at a scalar $\delta \in \Bbbk$; see [B, Theorem 1.4]. We will not discuss this specialization further here, but note that our arguments can be carried out in $\mathcal{H}eis_k(\delta)$ in exactly the same way as in $\mathcal{H}eis_k$. Consequently, Theorem 1.1 remains true when $\mathcal{H}eis_k$ is replaced by $\mathcal{H}eis_k(\delta)$. The specialized version with $k = -1, \delta = 0$ or with $k < 0, \delta \in \mathbb{Z}$ proves the original conjectures from [Kh] and [MS], respectively.

The main new ingredient needed to prove Theorem 1.1 is to show that γ_k is *surjective*. We do this by combining the strategy proposed by Khovanov in [Kh, Section 5] with one additional general result about Grothendieck groups; see Theorem 2.2. This additional result is well known (and easy to prove) in the setting of finite-dimensional algebras. However, we need it here for algebras that are not finite-dimensional and, at this level of generality, we actually could not find it elsewhere in the literature. Our proof exploits a crucial splitting hypothesis as a substitute for finite-dimensionality.

We also prove the following theorem, which categorifies the relation (1.6). An analogous result categorifying the commutation relations between h^+_m and h^-_n was recorded in [MS, Proposition 4.3], where it was used to construct the homomorphism γ_k in the first place. In our proof of Theorem 1.1 explained in Section 7, we give a new approach to the construction of γ_k, thereby making our arguments completely independent of *loc. cit.*. We are then able to exploit Theorem 1.1 to give a considerably simplified proof of the categorical relations; see Section 8.

Theorem 1.2. *In* $\mathrm{Kar}(\mathcal{H}eis_k)$*, there are distinguished isomorphisms*

$$H^+_m \otimes H^+_n \cong H^+_n \otimes H^+_m, \qquad H^+_m \otimes E^-_n \cong \bigoplus_{r=0}^{\min(m,n)} \bigoplus_{\lambda \in \mathcal{P}_{r,k}} E^-_{n-r} \otimes H^+_{m-r} \quad \text{if } k \geq 0,$$

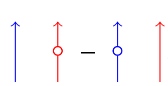
$$E^-_m \otimes E^-_n \cong E^-_n \otimes E^-_m, \qquad E^-_m \otimes H^+_n \cong \bigoplus_{r=0}^{\min(m,n)} \bigoplus_{\lambda \in \mathcal{P}_{r,-k}} H^+_{n-r} \otimes E^-_{m-r} \quad \text{if } k \leq 0,$$

where $\mathcal{P}_{r,n}$ *denotes the set of all partitions whose Young diagram fits into an* $r \times (n-r)$ *rectangle.*

The other key ingredient making this new approach possible is a strict monoidal functor

$$\Delta_{l|m} : \mathrm{Kar}(\mathcal{H}eis_k) \to \mathrm{Kar}(\mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m) \tag{1.9}$$

for $k = l + m$; see Theorem 5.3. Here, $- \odot -$ denotes symmetric product of strict monoidal categories (see Section 3 for the definition), and $\mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m$ is the localization of $\mathcal{H}eis_l \odot \mathcal{H}eis_m$ at the morphism

 (1.10)

where the left (blue) string comes from $\mathcal{H}eis_l$ and the right (red) string comes from $\mathcal{H}eis_m$. The following explains how $\Delta_{l|m}$ categorifies the comultiplication $\delta_{l|m}$ from (1.7).

Theorem 1.3. *For any* $k = l + m$*, there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Heis}_k \arrow[r, "\delta_{l|m}"] \arrow[d, hook, two heads, "\gamma_k"'] & \mathrm{Heis}_l \otimes_{\mathbb{Z}} \mathrm{Heis}_m \arrow[d, hook, two heads, "\gamma_l \otimes \gamma_m"] \\
 & K_0(\mathrm{Kar}(\mathcal{H}eis_l)) \otimes_{\mathbb{Z}} K_0(\mathrm{Kar}(\mathcal{H}eis_m)) \arrow[d, "\epsilon_{l|m}"] \\
K_0(\mathrm{Kar}(\mathcal{H}eis_k)) \arrow[r, "{[\Delta_{l|m}]}"'] & K_0(\mathrm{Kar}(\mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m)),
\end{tikzcd}$$

where $\epsilon_{l|m}$ is the ring homomorphism induced by the canonical functors from $\mathcal{H}eis_l$ and $\mathcal{H}eis_m$ to $\mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m$.

The categorical comultiplication $\Delta_{l|m}$ allows one to take tensor products of Heisenberg module categories provided that the morphism (1.10) acts invertibly (that is, there is no overlap in the spectrum of the red and blue dots). In Section 6, we give another application of this principle, namely, an efficient new proof of the basis theorem for morphism spaces in $\mathcal{H}eis_k$ from [B, Theorem 1.6] (where it is proved by invoking results of [Kh, MS] when $k < 0$ and [BCNR] when $k = 0$). The same general idea was first used in [W], and can also be adapted to other diagrammatic monoidal categories of a similar nature where bases are presently unavailable, including Frobenius/quantum analogs of the Heisenberg category and super Kac-Moody 2-categories. This will be developed elsewhere, e.g., see [BSW].

2. A general result about Grothendieck groups

In this section, until the final paragraph, all rings and modules are assumed to be unital. For a ring R, we let $K_0(R)$ denote the split Grothendieck group of the category R-pmod of finitely generated projective left R-modules. By definition (e.g., see [R, Definition 1.1.5]), this is the group completion of the commutative monoid consisting of isomorphism classes of finitely generated projectives with respect to the operation + induced by taking direct sums of modules. We write $[P]$ for the image of the isomorphism class of $P \in R$-pmod in $K_0(R)$. According to the definition of group completion, any element of $K_0(R)$ can be written in the form $[P] - [P']$ for $P, P' \in R$-pmod. Furthermore $[P] - [P'] = 0$ in $K_0(R)$ if and only if $P \oplus Q \cong P' \oplus Q$ for some $Q \in R$-pmod. Since Q is finitely generated and projective, it is a direct summand of a free module of finite rank. In other words, there exists $Q' \in R$-pmod and $n \geq 0$ such that $Q \oplus Q' \cong R^n$. Hence:

$$[P] - [P'] = 0 \text{ in } K_0(R) \iff P \oplus R^n \cong P' \oplus R^n \text{ for some } n \geq 0. \tag{2.1}$$

We say that R is *weakly cancellative* if we have that $[P] = 0 \Rightarrow P = 0$ for all $P \in R$-pmod; equivalently, $P \oplus R^n \cong R^n \Rightarrow P = 0$.

Lemma 2.1. *If R is finitely generated as a module over its center then R is weakly cancellative.*

Proof. Suppose that $P \oplus R^n \cong R^n$ for some non-zero $P \in R$-pmod. Since P is finitely generated over the center Z, the Nakayama lemma implies that the quotient $P/\mathfrak{m}P$ is non-zero for some maximal ideal $\mathfrak{m}$ of Z. Then we have that $P/\mathfrak{m}P \oplus (R/\mathfrak{m}R)^n \cong (R/\mathfrak{m}R)^n$ as $R/\mathfrak{m}R$-modules, hence, as $Z/\mathfrak{m}$-vector spaces. This is clearly impossible by dimension considerations. □

Suppose R and S are rings, and M is an (S, R)-bimodule that is finitely generated and projective as a left S-module. Then we have the induced functor

$$F : R\text{-pmod} \to S\text{-pmod}, \quad P \mapsto M \otimes_R P,$$

which induces a homomorphism of Abelian groups $[F] : K_0(R) \to K_0(S)$. The main result in this section is as follows.

Theorem 2.2. *Suppose R is a ring and $e \in R$ is an idempotent. Let $S := R/ReR$ and suppose that there exists a unital ring homomorphism $\sigma : S \to R$ such that $\pi \circ \sigma = \mathrm{id}_S$, where $\pi : R \to S$ is the quotient map. Then there is a split short exact sequence of Abelian groups*

$$0 \longrightarrow K_0(eRe) \xrightarrow{\phi} K_0(R) \xrightarrow{\psi} K_0(S) \longrightarrow 0, \tag{2.2}$$

where $\phi([P]) := [Re \otimes_{eRe} P]$ and $\psi([Q]) := [S \otimes_R Q]$. Moreover, R is weakly cancellative if and only if both eRe and S are weakly cancellative.

The proof will be carried out in the remainder of the section via a series of lemmas. We begin with some elementary remarks. First, the map ϕ is well defined since the (R, eRe)-bimodule Re is finitely generated and projective as a left R-module. Similarly, the map ψ is well defined since the (S,R)-bimodule S is finitely generated and projective as a left S-module. We may denote this bimodule also by S_π to make it clear that the right R-module structure is defined via the homomorphism $\pi : R \to S$. Similarly, we have the (R,S)-bimodule R_σ, which is the left regular R-module R with right action of S defined by $rs := r\sigma(s)$. Note that

$$S_\pi \otimes_R R_\sigma \cong S_{\pi\circ\sigma} = S \tag{2.3}$$

as an (S,S)-bimodule.

Lemma 2.3. *The map ψ from (2.2) is a split surjection.*

Proof. Since R_σ is finitely generated and projective as a left R-module, we get a well-defined map $\theta : K_0(S) \to K_0(R)$, $[P] \mapsto [R_\sigma \otimes_S P]$. Then the identity (2.3) implies that $\psi \circ \theta = \mathrm{id}$. □

Lemma 2.4. *The map ϕ is injective.*

Proof. As noted above, any element in $K_0(eRe)$ can be written in the form $[P] - [P']$ for some $P, P' \in eRe$-pmod. Suppose $[P]-[P'] \in \ker(\phi)$. Then we have that $[Re\otimes_{eRe}P]-[Re\otimes_{eRe}P'] = 0$, so by (2.1) there exists $n \in \mathbb{N}$ such that there is an isomorphism

$$\theta : Re \otimes_{eRe} P \oplus R^n \to Re \otimes_{eRe} P' \oplus R^n.$$

Writing maps on the right, θ can be represented by right multiplication by an invertible 2×2 matrix $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ for $A : Re\otimes_{eRe} P \to Re\otimes_{eRe} P'$, a row vector $B : Re\otimes_{eRe} P \to R^n$, a column vector $C : R^n \to Re \otimes_{eRe} P'$ and an $n \times n$ matrix $D \in M_n(R)$. The image $\pi(D)$ in $M_n(S)$ is invertible, so, with loss of generality, we can assume $\pi(D)$ is the identity matrix I_n. Then we have that $D = I_n - \sum_{k=1}^m A_k e B_k$ for some $m \geq 1$ and $A_k, B_k \in M_n(R)$. Consider the homomorphism

$$\theta' : Re \otimes_{eRe} P \oplus R^n \oplus (Re)^n \oplus \cdots \oplus (Re)^n \to Re \otimes_{eRe} P' \oplus R^n \oplus (Re)^n \oplus \cdots \oplus (Re)^n$$

(where there are m summands $(Re)^n$ on each side) defined by right multiplication by the matrix

$$X := \begin{bmatrix} A & B & 0 & 0 & \cdots & 0 \\ C & I_n & A_1e & A_2e & \cdots & A_me \\ 0 & eB_1 & eI_n & 0 & \cdots & 0 \\ 0 & eB_2 & 0 & eI_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & eB_m & 0 & 0 & \cdots & eI_n \end{bmatrix}.$$

By some obvious elementary row operations, the matrix X can be transformed into the invertible matrix

$$\begin{bmatrix} A & B & 0 & 0 & \cdots & 0 \\ C & D & 0 & 0 & \cdots & 0 \\ 0 & eB_1 & eI_n & 0 & \cdots & 0 \\ 0 & eB_2 & 0 & eI_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & eB_m & 0 & 0 & \cdots & eI_n \end{bmatrix}.$$

It follows that the matrix X is invertible. On the other hand, by some other elementary row and column operations, the matrix X can be transformed into a matrix of the form

$$\begin{bmatrix} Y_{1,1} & 0 & Y_{1,2} & Y_{1,3} & \cdots & Y_{1,m+1} \\ 0 & I_n & 0 & 0 & \cdots & 0 \\ Y_{2,1} & 0 & Y_{2,2} & Y_{2,3} & \cdots & Y_{2,m+1} \\ Y_{3,1} & 0 & Y_{3,2} & Y_{3,3} & \cdots & Y_{3,m+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{m+1,1} & 0 & Y_{m+1,2} & Y_{m+1,3} & \cdots & Y_{m+1,m+1} \end{bmatrix}.$$

This produces an invertible matrix $Y = (Y_{i,j})_{i,j=1,\dots,m+1}$ such that right multiplication by Y defines an isomorphism

$$\theta'' \colon Re \otimes_{eRe} P \oplus (Re)^n \oplus \cdots \oplus (Re)^n \to Re \otimes_{eRe} P' \oplus (Re)^n \oplus \cdots \oplus (Re)^n.$$

Finally, we restrict θ'' to $eRe \otimes_{eRe} P \oplus (eRe)^n \oplus \cdots \oplus (eRe)^n$, noting that $eRe \otimes_{eRe} P \cong P$ and $eRe \otimes_{eRe} P' \cong P'$, to obtain an isomorphism of eRe-modules $P \oplus (eRe)^{mn} \cong P' \oplus (eRe)^{mn}$. Hence, $[P] - [P'] = 0$ in $K_0(eRe)$ by (2.1). □

Lemma 2.5. *We have that* $\psi \circ \phi = 0$.

Proof. For any right R-module M, the multiplication map is an isomorphism $M \otimes_R Re \cong Me$. Applying this to $M = S_\pi$, we see that $S_\pi \otimes_R Re \cong (S_\pi)e$, which is zero as $\pi(e) = 0$. The map $\psi \circ \phi$ is defined by tensoring with this bimodule. □

Lemma 2.6. *If* $P \in R$-pmod *and* $S_\pi \otimes_R P = 0$*, then* $P \cong Re \otimes_{eRe} V$ *for some* $V \in eRe$-pmod*.*

Proof. Suppose $P \in R$-pmod and $S_\pi \otimes_R P = 0$. Let $V := eP$, which is naturally an eRe-module. Consider the homomorphism of R-modules

$$\mu \colon Re \otimes_{eRe} V \to P, \quad ae \otimes v \mapsto aev.$$

Since $0 = S_\pi \otimes_R P = (R/ReR) \otimes_R P \cong P/ReP$, it follows that $ReP = P$. Hence, μ is surjective. Since P is projective as a left R-module, the map μ splits, so we have a homomorphism of R-modules $\tau \colon P \to Re \otimes_{eRe} V$ such that $\mu \circ \tau = \mathrm{id}_P$. Restricting, we have

$$V = eP \xrightarrow{\tau} eRe \otimes_{eRe} V \underset{\cong}{\xrightarrow{\mu}} V.$$

In other words, $\tau|_V$ splits the isomorphism $\mu|_{eRe \otimes_{eRe} V}$ and hence must be its inverse. Thus $e \otimes V \subseteq \operatorname{im} \tau$. It follows that τ is surjective, hence, an isomorphism. We have now shown that $P \cong Re \otimes_{eRe} V$ as R-modules. It remains to show that V is finitely generated and projective.

Since $P \in R$-pmod, we can choose elements $p_1, \dots, p_m$ that generate P as an R-module. As noted above, we have $P = ReP$. Hence, for each $i = 1, \dots, m$, we can write

$$p_i = \sum_{j=1}^{n_i} a_{i,j} e q_{i,j}$$

for some $n_i \geq 0$, $a_{i,j} \in R$ and $q_{i,j} \in P$. The elements $\{eq_{i,j} \mid i = 1, \dots, m, j = 1, \dots, n_i\}$ generate V as an eRe-module. So V is finitely generated.

To see that V is projective, suppose we have a surjective homomorphism of eRe-modules $\theta \colon U \twoheadrightarrow V$. Then we have an induced surjective homomorphism of R-modules

$$\mathrm{id} \otimes \theta \colon Re \otimes_{eRe} U \twoheadrightarrow Re \otimes_{eRe} V \cong P.$$

Since P is projective, this map splits. So we have a homomorphism of R-modules

$$\xi \colon Re \otimes_{eRe} V \to Re \otimes_{eRe} U$$

such that $(\mathrm{id} \otimes \theta) \circ \xi = \mathrm{id}_{Re \otimes_{eRe} V}$. From this, we see that the restriction $\xi|_{eRe \otimes_{eRe} V}$ splits the restriction $(\mathrm{id} \otimes \theta)|_{eRe \otimes_{eRe} U} : eRe \otimes_{eRe} U \to eRe \otimes_{eRe} V$. Under the natural isomorphisms $eRe \otimes_{eRe} U \cong U$ and $eRe \otimes_{eRe} V \cong V$, the map $(\mathrm{id} \otimes \theta)|_{eRe \otimes_{eRe} U}$ corresponds to θ. So θ splits. □

Lemma 2.7. *Suppose* $P \in R$-pmod *and let* $Q := S_\pi \otimes_R P$*. There exists* $V \in eRe$-pmod *and* $n \geq 0$ *such that*

$$R_\sigma \otimes_S Q \oplus (Re)^n \cong P \oplus Re \otimes_{eRe} V$$

as R-modules.

Proof. By Frobenius reciprocity, we have a natural isomorphism

$$\mathrm{Hom}_R(R_\sigma \otimes_S Q, P) \cong \mathrm{Hom}_S(Q, \mathrm{Hom}_R(R_\sigma, P)). \tag{2.4}$$

Moreover, $\mathrm{Hom}_R(R_\sigma, P) \cong {}_\sigma P$, meaning the left R-module P viewed as an S-module via the map $\sigma\colon S \to R$. Since $Q = (R/ReR) \otimes_R P \cong P/ReP$, we have a short exact sequence

$$0 \longrightarrow {}_\sigma ReP \longrightarrow {}_\sigma P \longrightarrow Q \longrightarrow 0.$$

Since Q is projective as an S-module, we have a splitting $\tau\colon Q \to {}_\sigma P$. Let $\nu\colon R_\sigma \otimes_S Q \to P$ be the R-module homomorphism corresponding to τ under (2.4), i.e., $\nu(a \otimes q) = a\tau(q)$.

As S-modules, we have ${}_\sigma P \cong {}_\sigma ReP \oplus \mathrm{im}\,\tau$. Thus, since $\mathrm{im}\,\nu \supseteq \mathrm{im}\,\tau$, we have $P = (\mathrm{im}\,\nu) + ReP$. Let $p_1, \dots, p_m$ generate P as an R-module and, for $i = 1, \dots, m$, write

$$p_i = \nu(u_i) + \sum_{j=1}^{n} a_{i,j} q_j,$$

for some $n \geq 0$, $u_i \in R_\sigma \otimes_S Q$, $a_{i,j} \in Re$ and $q_j \in eP$. Then the map

$$\theta\colon R_\sigma \otimes_S Q \oplus (Re)^n \to P, \quad (u, b_1, \dots, b_n) \mapsto \nu(u) + \sum_{j=1}^{n} b_j q_j,$$

is a surjective R-module homomorphism. Since P is projective as an R-module, this map splits. So we have

$$R_\sigma \otimes_S Q \oplus (Re)^n \cong P \oplus (\ker\theta).$$

When we apply the functor $S_\pi \otimes_R -$ to the split short exact sequence

$$0 \longrightarrow \ker\theta \longrightarrow R_\sigma \otimes_S Q \oplus (Re)^n \xrightarrow{\theta} P \longrightarrow 0,$$

we obtain a short exact sequence

$$0 \longrightarrow S_\pi \otimes_R \ker\theta \longrightarrow S_\pi \otimes_R R_\sigma \otimes_S Q \xrightarrow{\mathrm{id}_S \otimes\theta} Q \longrightarrow 0.$$

The composite of $\mathrm{id}_S \otimes\theta$ and the isomorphism $Q \xrightarrow{\sim} S_\pi \otimes_R R_\sigma \otimes_S Q$ from (2.3) is the identity id_Q, hence, $\mathrm{id}_S \otimes\theta$ is an isomorphism. This implies that $S_\pi \otimes_R \ker\theta = 0$. Finally, we apply Lemma 2.6 to $\ker\theta \in R$-pmod to deduce that $\ker\theta \cong Re \otimes_{eRe} V$ for some $V \in eRe$-pmod. □

Lemma 2.8. *We have* $\ker\psi \subseteq \mathrm{im}\,\phi$.

Proof. Consider an arbitrary element $[P] - [P'] \in \ker\psi$, where $P, P' \in R$-pmod. In $K_0(S)$, we have that $[Q] - [Q'] = 0$ where $Q := S_\pi \otimes_R P$ and $Q' := S_\pi \otimes_R P'$. By (2.1), we can assume (replacing P and P' by $P \oplus R^n$ and $P' \oplus R^n$ for some $n \geq 0$) that $Q \cong Q'$ as S-modules. By the second isomorphism from (2.3), we have that $R_\sigma \otimes_S Q \cong R_{\sigma\circ\pi} \otimes_R P$ and $R_\sigma \otimes_S Q' \cong R_{\sigma\circ\pi} \otimes_R P'$. Applying Lemma 2.7 twice, we get $V, V' \in eRe$-pmod and $n, n' \geq 0$ such that

$$(Re)^n \oplus R_\sigma \otimes_S Q \cong P \oplus Re \otimes_{eRe} V, \qquad (Re)^{n'} \oplus R_\sigma \otimes_S Q' \cong P' \oplus Re \otimes_{eRe} V'.$$

Since $R_\sigma \otimes_S Q \cong R_\sigma \otimes_S Q'$ as R-modules, we deduce that

$$[P] - [P'] = (n - n')[Re] - [Re \otimes_{eRe} V] + [Re \otimes_{eRe} V'],$$

which belongs to $\mathrm{im}\,\phi$. □

Proof of Theorem 2.2. The fact that (2.2) is split exact follows from Lemmas 2.3, 2.4, 2.5 and 2.8. For the final part, suppose first that eRe and S are both weakly cancellative. Take $P \in R$-pmod with $[P] = 0$. Then $[S_\pi \otimes_R P] = 0$ which implies that $S_\pi \otimes_R P = 0$. Applying Lemma 2.6, we deduce that $P \cong Re \otimes_{eRe} V$ for some $V \in eRe$-pmod. As $[Re \otimes_{eRe} V] = 0$, Lemma 2.4 now gives that $[V] = 0$. Hence, $V = 0$, so $P = 0$ too.

Conversely, suppose that R is weakly cancellative. Take $V \in eRe$-pmod with $[V] = 0$. Then $[Re \otimes_{eRe} V] = 0$ which implies that $Re \otimes_{eRe} V = 0$. Then multiply by the idempotent e to get that $V \cong eRe \otimes_{eRe} V = 0$. Finally, take $Q \in S$-pmod with $[Q] = 0$. Then $[S_\pi \otimes_R Q] = 0$ which implies that $S_\pi \otimes_S Q = 0$. Hence, $Q \cong R_\sigma \otimes_R S_\pi \otimes_S Q = 0$. □

We are going to be working in the remainder of the article with (usually monoidal) $\Bbbk$-linear categories instead of rings. The data of a $\Bbbk$-linear category $\mathcal{A}$ is the same as the data of a *locally unital algebra*, i.e., an associative (but not necessarily unital) $\Bbbk$-algebra A equipped with a system of mutually orthogonal idempotents $\{1_X \mid X \in \mathbb{A}\}$ such that

$$A = \bigoplus_{X,Y\in\mathbb{A}} 1_Y A 1_X. \tag{2.5}$$

Under this identification, $\mathbb{A}$ is the object set of $\mathcal{A}$, the idempotent 1_X is the identity endomorphism of object X, $1_Y A 1_X := \mathrm{Hom}_{\mathcal{A}}(X, Y)$, and multiplication in A is induced by composition in $\mathcal{A}$. By a module over such a locally unital algebra, we mean a left module V as usual such that $V = \bigoplus_{X\in\mathbb{A}} 1_X V$. This is just the same data as a $\Bbbk$-linear functor from $\mathcal{A}$ to vector spaces. Let A-pmod be the category of finitely generated projective A-modules. Then the Yoneda lemma implies that there is a *contravariant* equivalence of categories

$$\mathrm{Kar}(\mathcal{A}) \to A\text{-pmod} \tag{2.6}$$

sending an object $X \in \mathcal{A}$ to the left ideal $A1_X$, and a morphism $f : X \to Y$ to the homomorphism $A1_Y \to A1_X$ defined by right multiplication. We get induced a canonical isomorphism

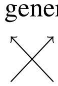
$$K_0(\mathrm{Kar}(\mathcal{A})) \cong K_0(A), \tag{2.7}$$

where $K_0(A)$ denotes the split Grothendieck group of A-pmod. Providing A is actually unital, i.e., $\mathcal{A}$ has only finitely many non-zero objects, Theorem 2.2 can then be applied in this situation.

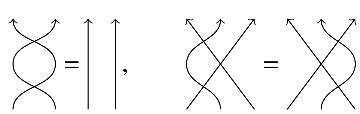
3. Categorification of symmetric functions

It is well known that the ring $\mathrm{Sym}_{\mathbb{Z}}$ of symmetric functions is categorified by the representations of the symmetric groups $\mathfrak{S}_n$ for all n. In this section, we are going to reformulate this classical result in terms of monoidal categories. This will give us the opportunity to introduce language which will be essential later on.

Let $\mathcal{S}ym$ be the free strict $\Bbbk$-linear symmetric monoidal category generated by one object. This has a very simple monoidal presentation in terms of the string calculus for morphisms in strict monoidal categories; see e.g. [TV, Chapter 1]. Our general conventions for this are different from *loc. cit.*: we represent the horizontal composition $f \otimes g$ (resp., vertical composition $f \circ g$) of morphisms f and g diagrammatically by drawing f to the left of g (resp., drawing f above g). We denote the unit object by $\mathbb{1}$ and its identity endomorphism by $1_{\mathbb{1}}$. Then, $\mathcal{S}ym$ is the strict $\Bbbk$-linear monoidal category generated by one object $\uparrow$ and one morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$

subject to the relations

 (3.1)

The objects of $\mathcal{S}ym$ are the tensor powers $\uparrow^{\otimes n}$ of the generating object for $n \in \mathbb{N}$. There are no non-zero morphisms between $\uparrow^{\otimes m}$ and $\uparrow^{\otimes n}$ for $m \neq n$. Moreover, there is an algebra isomorphism

$$\imath_n : \Bbbk\mathfrak{S}_n \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}ym}(\uparrow^{\otimes n}) \tag{3.2}$$

sending the ith basic transposition s_i to the crossing of the ith and $(i+1)$th strings (remembering that we number strings $1, 2, \ldots$ from right to left). Thus $\mathcal{S}ym$ assembles the group algebras of all the symmetric groups into one convenient package.

Now we can use the equivalence (2.6) and the isomorphism (2.7) to translate the well-known representation theory of symmetric groups into statements about $\mathcal{S}ym$. Since we are in characteristic zero, Maschke's theorem implies that the additive Karoubi envelope $\mathrm{Kar}(\mathcal{S}ym)$

is a semisimple Abelian category. For $\lambda \in \mathcal{P}$ with $|\lambda| = n$, the Specht module $S(\lambda) = (\Bbbk\mathfrak{S}_n)e_\lambda$ corresponds to the indecomposable object $S_\lambda := (\uparrow^{\otimes n}, \iota_n(e_\lambda)) \in \mathrm{Kar}(\mathcal{S}ym)$. We set $H_n := S_{(n)}$ and $E_n := S_{(1^n)}$ for short. Then we see that the classes $\left\{[S_\lambda] \,\middle|\, \lambda \in \mathcal{P}\right\}$ give a basis for $K_0(\mathrm{Kar}(\mathcal{S}ym))$ as a free $\mathbb{Z}$-module. Moreover, since taking tensor products of idempotents in $\mathrm{Kar}(\mathcal{S}ym)$ corresponds to the induction product at the level of $\Bbbk\mathfrak{S}_n$-modules, the Littlewood-Richardson rule implies that there is a ring isomorphism

$$\gamma : \mathrm{Sym}_{\mathbb{Z}} \xrightarrow{\sim} K_0(\mathrm{Kar}(\mathcal{S}ym)), \qquad s_\lambda \mapsto [S_\lambda], \quad h_n \mapsto [H_n], \quad e_n \mapsto [E_n]. \tag{3.3}$$

Thus $\mathcal{S}ym$ categorifies the ring of symmetric functions.

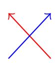
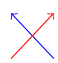
In the remainder of the section, we are going to explain how to categorify the comultiplication (1.2) on $\mathrm{Sym}_{\mathbb{Z}}$. The usual way to do this is by considering the restriction functors from $\mathfrak{S}_n$ to $\mathfrak{S}_r \times \mathfrak{S}_{n-r}$ for all $0 \leq r \leq n$. We are going to formulate the result instead in terms of a monoidal functor on $\mathcal{S}ym$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two strict $\Bbbk$-linear monoidal categories. We can obviously form their free product $\mathcal{C} \circledast \mathcal{D}$ as a strict $\Bbbk$-linear monoidal category. Thus there are canonical strict $\Bbbk$-linear monoidal functors from each of $\mathcal{C}$ and $\mathcal{D}$ to $\mathcal{C} \circledast D$, and $\mathcal{C} \circledast \mathcal{D}$ is universal amongst all strict $\Bbbk$-linear monoidal categories that are targets of such a pair of functors. The *symmetric product* $\mathcal{C} \odot \mathcal{D}$ is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{C} \circledast \mathcal{D}$ by adjoining isomorphisms $\sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ for each pair of objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, subject to the relations

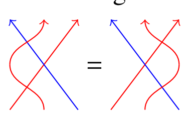
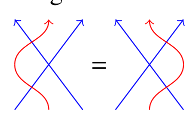
$$\begin{aligned} \sigma_{X_1 \otimes X_2, Y} &= (\sigma_{X_1,Y} \otimes 1_{X_2}) \circ (1_{X_1} \otimes \sigma_{X_2,Y}), & \sigma_{X_2,Y} \circ (f \otimes 1_Y) &= (1_Y \otimes f) \circ \sigma_{X_1,Y}, \\ \sigma_{X, Y_1 \otimes Y_2} &= (1_{Y_1} \otimes \sigma_{X,Y_2}) \circ (\sigma_{X,Y_1} \otimes 1_{Y_2}), & \sigma_{X,Y_2} \circ (1_X \otimes g) &= (g \otimes 1_X) \circ \sigma_{X,Y_1} \end{aligned}$$

for all $X, X_1, X_2 \in \mathcal{C}$, $Y, Y_1, Y_2 \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$, $g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)$.

The symmetric product $\mathcal{S}ym \odot \mathcal{S}ym$ of two copies of $\mathcal{S}ym$ is the free strict $\Bbbk$-linear symmetric monoidal category generated by two objects. Diagrammatically it is convenient to use different colors, denoting the symmetric product instead by $\mathcal{S}ym \odot \mathcal{S}ym$ and using the color blue (resp., red) for objects and morphisms in the first (resp., second) copy of $\mathcal{S}ym$. Morphisms may then be represented by linear combinations of string diagrams colored both blue and red. In these diagrams, as well as the one-color crossings that are the generating morphisms of Sym and Sym, we have the additional two-color crossings

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow,$  $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow,$ (3.4)

which are mutual inverses. The definition of symmetric product gives braid-like relations allowing all one-color crossings to be commuted across strings of the other color, for example:

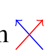
 ,  . (3.5)

For $0 \leq r \leq n$ let $\mathcal{P}_{r,n}$ denote the set of size $\binom{n}{r}$ consisting of tuples $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ such that $n - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Let $\min_{r,n}$ (resp., $\max_{r,n}$) be the element $\lambda \in \mathcal{P}_{r,n}$ with $\lambda_1 = \cdots = \lambda_r = 0$ (resp., $\lambda_1 = \cdots = \lambda_r = n - r$). For any $\lambda \in \mathcal{P}_{r,n}$, we let

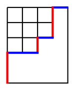
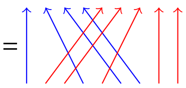
$$\uparrow^{\otimes\lambda} := \uparrow^{\otimes(n-r-\lambda_1)} \otimes \uparrow \otimes \uparrow^{\otimes(\lambda_1-\lambda_2)} \otimes \uparrow \otimes \cdots \otimes \uparrow \otimes \uparrow^{\otimes\lambda_r} \in \mathcal{S}ym \odot \mathcal{S}ym; \tag{3.6}$$

in particular, $\uparrow^{\otimes \min_{r,n}} = \uparrow^{\otimes(n-r)} \otimes \uparrow^{\otimes r}$ and $\uparrow^{\otimes \max_{r,n}} = \uparrow^{\otimes r} \otimes \uparrow^{\otimes(n-r)}$. In this way, $\mathcal{P}_{r,n}$ labels the objects of $\mathcal{S}ym \odot \mathcal{S}ym$ obtained by tensoring r generators $\uparrow$ and $(n-r)$ generators $\uparrow$ in some order. We denote the identity endomorphism of $\uparrow^{\otimes\lambda}$ simply by 1_λ. There is also a unique isomorphism

$$\sigma_\lambda : \uparrow^{\otimes\lambda} \xrightarrow{\sim} \uparrow^{\otimes \min_{r,n}} \tag{3.7}$$

whose diagram only involves crossings of the form ; in particular, $\sigma_{\min_{r,n}} = 1_{\min_{r,n}}$. To make sense of these definitions, one can represent an element of $\mathcal{P}_{r,n}$ by a Young diagram with λ_i

boxes on its ith row drawn inside an $r \times (n-r)$-rectangle. Then $\uparrow^{\otimes\lambda}$ may be seen by walking southwest along the boundary of the diagram; for example, $(3,3,2,0,0) \in \mathcal{P}_{5,9}$ is

$\lambda =$  , $\qquad \uparrow^{\otimes\lambda} = \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow, \qquad$ $\sigma_\lambda =$  .

We will often identify the group algebra $\Bbbk\mathfrak{S}_r \otimes_\Bbbk \Bbbk\mathfrak{S}_{n-r}$ of $\mathfrak{S}_r \times \mathfrak{S}_{n-r}$ with a subalgebra of $\Bbbk\mathfrak{S}_n$ so that $s_i \otimes 1 \leftrightarrow s_i$ and $1 \otimes s_j \leftrightarrow s_{r+j}$. There is an algebra isomorphism

$$\imath_{r,n} : \Bbbk\mathfrak{S}_r \otimes_\Bbbk \Bbbk\mathfrak{S}_{n-r} \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}ym \odot \mathcal{S}ym}\left(\uparrow^{\otimes(n-r)} \otimes \uparrow^{\otimes r}\right) \tag{3.8}$$

sending $s_i = s_i \otimes 1$ to the crossing of the ith and $(i+1)$th red strings and $s_{r+j} = 1 \otimes s_j$ to the crossing of the jth and $(j+1)$th blue strings. Combining this isomorphism with the elements $\left\{\sigma_\lambda^{-1} \circ \sigma_\mu \,\middle|\, \lambda, \mu \in \mathcal{P}_{r,n}\right\}$, which give the matrix units, we see that

$$\mathrm{End}_{\mathrm{Add}(\mathcal{S}ym \odot \mathcal{S}ym)}\left((\uparrow \oplus \uparrow)^{\otimes n}\right) \cong \bigoplus_{r=0}^{n} \mathrm{Mat}_{\binom{n}{r}}\left(\Bbbk\mathfrak{S}_r \otimes_\Bbbk \Bbbk\mathfrak{S}_{n-r}\right). \tag{3.9}$$

Using (2.6)–(2.7) too, we conclude that $K_0(\mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym)) \cong \mathrm{Sym}_\mathbb{Z} \otimes_\mathbb{Z} \mathrm{Sym}_\mathbb{Z}$. An explicit isomorphism is given by the composition

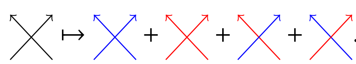
$$\mathrm{Sym}_\mathbb{Z} \otimes_\mathbb{Z} \mathrm{Sym}_\mathbb{Z} \xrightarrow{\gamma\otimes\gamma} K_0(\mathrm{Kar}(\mathcal{S}ym)) \otimes_\mathbb{Z} K_0(\mathrm{Kar}(\mathcal{S}ym)) \xrightarrow{\epsilon} K_0(\mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym))$$

where the second map ϵ is induced by the inclusions of $\mathcal{S}ym$ and $\mathcal{S}ym$ into $\mathcal{S}ym \odot \mathcal{S}ym$.

Now we are ready to define a strict $\Bbbk$-linear monoidal functor

$$\Delta : \mathcal{S}ym \to \mathrm{Add}(\mathcal{S}ym \odot \mathcal{S}ym) \tag{3.10}$$

by sending the generating object $\uparrow$ to $\uparrow \oplus \uparrow$, and defined on the generating morphism by

 (3.11)

The right-hand side of this, which is a 4×4 matrix in $\mathrm{End}_{\mathrm{Add}(\mathcal{S}ym\odot\mathcal{S}ym)}(\uparrow\otimes\uparrow\oplus\uparrow\otimes\uparrow\oplus\uparrow\otimes\uparrow\oplus\uparrow\otimes\uparrow)$, is the morphism defining the symmetric braiding on the object $\uparrow \oplus \uparrow$ of $\mathrm{Add}(\mathcal{S}ym \odot \mathcal{S}ym)$ with respect to its canonical symmetric monoidal structure as the additive envelope of the symmetric monoidal category $\mathcal{S}ym \odot \mathcal{S}ym$. The fact that Δ is well defined is immediate from the universal property of $\mathcal{S}ym$ as the free symmetric monoidal category on one object; alternatively, one can directly verify that the defining relations (3.1) are satisfied. To compute Δ on a more general diagram D, one just has to sum over all diagrams obtained from D by coloring the strings red or blue in all possible ways.

Remark 3.1. Similarly, there is a monoidal functor $\mathcal{S}ym \to \mathrm{Add}\,(\mathcal{S}ym \odot \mathcal{S}ym \odot \mathcal{S}ym)$ to the triple symmetric product which sends $\uparrow$ to $\uparrow \oplus \uparrow \oplus \uparrow$. Identifying $\mathcal{S}ym \odot \mathcal{S}ym \odot \mathcal{S}ym$ with $(\mathcal{S}ym \odot \mathcal{S}ym) \odot \mathcal{S}ym$ and $\mathcal{S}ym \odot (\mathcal{S}ym \odot \mathcal{S}ym)$, this agrees with both of the compositions $(\Delta \odot \mathrm{Id}) \circ \Delta$ and $(\mathrm{Id} \odot \Delta) \circ \Delta$. In other words, the categorical comultiplication is coassociative.

The functor Δ extends to a monoidal functor $\Delta : \mathrm{Kar}(\mathcal{S}ym) \to \mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym)$, which in turn induces $[\Delta] : K_0(\mathrm{Kar}(\mathcal{S}ym)) \to K_0(\mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym))$. Note that $[\Delta]$ is automatically a ring homomorphism; the analogous statement in the more traditional approach via restriction functors requires an application of the Mackey theorem at this point. We claim moreover that

$$\begin{tikzcd} \mathrm{Sym}_\mathbb{Z} \arrow[r, "\delta"] \arrow[d, hook, two heads, "\gamma"'] & \mathrm{Sym}_\mathbb{Z} \otimes_\mathbb{Z} \mathrm{Sym}_\mathbb{Z} \arrow[d, hook, two heads, "\epsilon\circ\gamma\otimes\gamma"] \\ K_0(\mathrm{Kar}(\mathcal{S}ym)) \arrow[r, "{[\Delta]}"'] & K_0(\mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym)) \end{tikzcd} \tag{3.12}$$

commutes, i.e., Δ categorifies the comultiplication δ on $\mathrm{Sym}_\mathbb{Z}$. This is a consequence of the following theorem, bearing in mind that the complete symmetric functions h_n generate $\mathrm{Sym}_\mathbb{Z}$.

Theorem 3.2. *For each $n \geq 0$, we have that*

$$\Delta(H_n) \cong \bigoplus_{r=0}^{n} H_{n-r} \otimes H_r, \qquad \Delta(E_n) \cong \bigoplus_{r=0}^{n} E_{n-r} \otimes E_r. \tag{3.13}$$

Proof. For the isomorphism involving H_n, it suffices to show that the idempotents $\Delta(\iota_n(e_{(n)}))$ and $\sum_{r=0}^{n} \iota_{r,n}(e_{(r)} \otimes e_{(n-r)})$ which define the objects $\Delta(H_n)$ and $\bigoplus_{r=0}^{n} H_{n-r} \otimes H_r$ are conjugate. Thus, we need to construct morphisms u and v in $\mathrm{Kar}(\mathcal{S}ym \odot \mathcal{S}ym)$ such that $u \circ v = \Delta(\iota_n(e_{(n)}))$ and $v \circ u = \sum_{r=0}^{n} \iota_{r,n}(e_{(r)} \otimes e_{(n-r)})$. To do this, notice for any $\lambda, \mu \in \mathcal{P}_{r,n}$ that

$$1_\mu \circ \Delta(\iota_n(e_{(n)})) \circ 1_\lambda = \binom{n}{r}^{-1} \sigma_\mu^{-1} \circ \iota_{r,n}(e_{(r)} \otimes e_{(n-r)}) \circ \sigma_\lambda.$$

It follows that $\Delta(\iota_n(e_{(n)})) = u \circ v$ where

$$u := \sum_{r=0}^{n} \binom{n}{r}^{-1} \sum_{\mu \in \mathcal{P}_{r,n}} \sigma_\mu^{-1} \circ \iota_{r,n}(e_{(r)} \otimes e_{(n-r)}), \qquad v := \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_{r,n}} \iota_{r,n}(e_{(r)} \otimes e_{(n-r)}) \circ \sigma_\lambda.$$

Finally it is easy to check that $v \circ u = \sum_{r=0}^{n} \iota_{r,n}(e_{(r)} \otimes e_{(n-r)})$.

To establish the isomorphism involving E_n, one needs to show instead that there are morphisms u and v such that $u \circ v = \Delta(\iota_n(e_{(1^n)}))$ and $v \circ u = \sum_{r=0}^{n} \iota_{r,n}(e_{(1^r)} \otimes e_{(1^{n-r})})$. These are given by similar formulae to the above, replacing $e_{(m)}$ by $e_{(1^m)}$ and σ_ν by $(-1)^{|\nu|}\sigma_\nu$ everywhere. ◻

4. The degenerate affine Hecke category

The *degenerate affine Hecke algebra AH_n* is the vector space $\Bbbk[x_1, \dots, x_n] \otimes_\Bbbk \Bbbk\mathfrak{S}_n$ viewed as an associative algebra with multiplication defined so that $\Bbbk[x_1, \dots, x_n]$ and $\Bbbk\mathfrak{S}_n$ are subalgebras, and in addition

$$s_i f = s_i(f)\, s_i + \partial_i(f) \tag{4.1}$$

for $f \in \Bbbk[x_1, \dots, x_n]$ and $i = 1, \dots, n-1$, where ∂_i is the *Demazure operator*

$$\partial_i(f) := \frac{f - s_i(f)}{x_{i+1} - x_i}. \tag{4.2}$$

Also recall that the *center* of AH_n is the subalgebra $\mathrm{Sym}_n := \Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n}$ of symmetric polynomials; see e.g. [Kl, Theorem 3.3.1]. The algebra AH_n is finitely generated as a Sym_n-module. The following theorem was proved by Khovanov in [Kh]; its proof uses the assumption that $\Bbbk$ is of characteristic zero in an essential way.

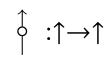
Theorem 4.1. *The inclusion $\Bbbk\mathfrak{S}_n \hookrightarrow AH_n$ induces an isomorphism $K_0(\Bbbk\mathfrak{S}_n) \xrightarrow{\sim} K_0(AH_n)$. More generally, the same assertion holds when $\Bbbk\mathfrak{S}_n$ is replaced with $\Bbbk\mathfrak{S}_{n_1} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathfrak{S}_{n_r}$ and AH_n is replaced with $AH_{n_1} \otimes_\Bbbk \cdots \otimes_\Bbbk AH_{n_r} \otimes_\Bbbk B$ for any $n_1, \dots, n_r \geq 0$ and any polynomial algebra B (possibly of infinite rank).*

Proof. This is explained in [Kh, Section 5.2]; see especially [Kh, (40)][1]. The argument in *loc. cit.* depends ultimately on a result of Quillen [Q, Theorem 7]. In order to be able to apply Quillen's result, one needs to know that the degenerate affine Hecke algebra AH_n is a filtered deformation of the smash product $\Bbbk[x_1, \dots, x_n]\#\mathfrak{S}_n$, which is a positively graded algebra with degree zero component given by the semisimple algebra $\Bbbk\mathfrak{S}_n$. When B is of infinite rank, one needs to know also that taking K_0 commutes with direct limits [R, Theorem 1.2.5]. ◻

The first part of this theorem implies that one can categorify the ring $\mathrm{Sym}_\mathbb{Z}$ using the algebra AH_n in place of $\Bbbk\mathfrak{S}_n$. Of course, we are going to translate this into the language of monoidal

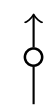
[1]This is (44) in the preprint version available at `https://arxiv.org/abs/1009.3295`.

categories. Let $\mathcal{AH}$ be the strict $\Bbbk$-linear monoidal category obtained from the category $\mathcal{S}ym$ from the previous section by adjoining an additional generating morphism



subject to the additional relations


(4.3)

In fact, in the presence of the quadratic relation in $\mathcal{S}ym$, the two relations in (4.3) are equivalent. We denote the ath power of  under vertical composition by labeling the dot with the multiplicity $a \in \mathbb{N}$. Just like for $\mathcal{S}ym$, there are no non-zero morphisms between $\uparrow^{\otimes m}$ and $\uparrow^{\otimes n}$ for $m \neq n$. Moreover, replacing (3.2), there is an algebra isomorphism

$$\imath_n : AH_n \xrightarrow{\sim} \mathrm{End}_{\mathcal{AH}}(\uparrow^{\otimes n}) \tag{4.4}$$

sending s_i to the crossing of the ith and $(i+1)$th strings and x_j to the dot on the jth string. Using (2.6)–(2.7) and Theorem 4.1, we deduce that the canonical monoidal embedding $\mathcal{S}ym \to \mathcal{AH}$ induces a ring isomorphism $K_0(\mathrm{Kar}(\mathcal{S}ym)) \xrightarrow{\sim} K_0(\mathrm{Kar}(\mathcal{AH}))$. Thus, we can reformulate (3.3): there is a ring isomorphism

$$\gamma : \mathrm{Sym}_{\mathbb{Z}} \xrightarrow{\sim} K_0(\mathrm{Kar}(\mathcal{AH})), \qquad s_\lambda \mapsto [S_\lambda], \quad h_n \mapsto [H_n], \quad e_n \mapsto [E_n], \tag{4.5}$$

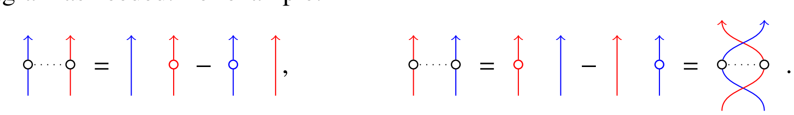
viewing S_λ, H_n and E_n now as objects of $\mathrm{Kar}(\mathcal{AH})$.

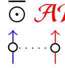
The next obvious question is whether the monoidal functor Δ from (3.10) can be upgraded from $\mathcal{S}ym$ to $\mathcal{AH}$ too. To do this, it turns out that we need to localize.

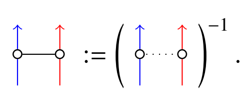
Consider the symmetric product $\mathcal{AH} \odot \mathcal{AH}$. This is generated by the objects and morphisms from two copies of $\mathcal{AH}$, one drawn in blue and the other in red, plus the additional two-color crossings as in (3.4). As well as (3.5), dots of one color commute across strings of the other:

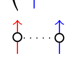

(4.6)

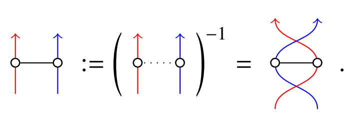
Given a diagram D representing a morphism in $\mathcal{AH} \odot \mathcal{AH}$ and two generic points in this diagram, one on a red string and the other on a blue string, we will denote the morphism represented by (D with an extra dot at the red point) − (D with an extra dot at the blue point) by joining the two points with a dotted line; this line may pass willy nilly through other strings in the diagram as needed. For example:


(4.7)

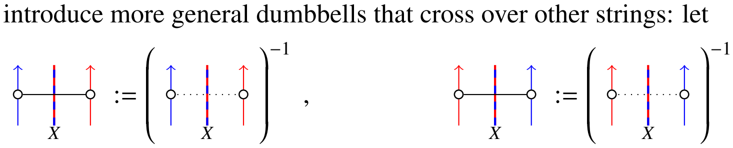
Let $\mathcal{AH} \mathbin{\bar{\odot}} \mathcal{AH}$ be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{AH} \odot \mathcal{AH}$ by localizing at . This means that we adjoin a two-sided inverse to this morphism, which we denote as a dumbbell


(4.8)

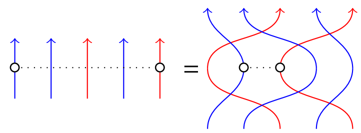
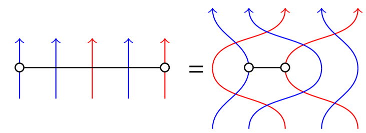
By the commuting relations, the morphism  is also invertible in $\mathcal{AH} \mathbin{\bar{\odot}} \mathcal{AH}$, with two-sided inverse



We can also introduce more general dumbbells that cross over other strings: let



for any object $X \in \mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH}$, where the two-colored vertical line represents 1_X. To see that this makes sense, one needs to prove that this morphism is indeed invertible; this follows easily from the commuting relations. For example, if $X = \uparrow \otimes \uparrow \otimes \uparrow$ then

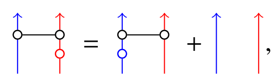
 with inverse 

Note also that $\mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH}$ has a monoidal involution

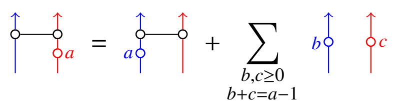
$$\mathrm{flip} : \mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH} \to \mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH} \tag{4.9}$$

which is defined on diagrams by switching the colors blue and red then multiplying by $(-1)^z$ where z is the total number of dumbbells in the picture.

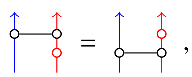
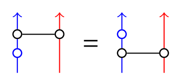
There are several other useful relations in $\mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH}$. Composing the definition (4.7) on the top with the dumbbell, we get that

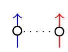


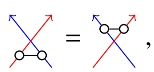
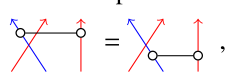
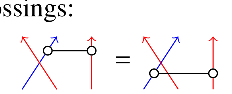
which gives a way to teleport dots across dumbbells modulo a correction term. More generally:

 (4.10)

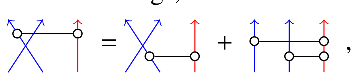
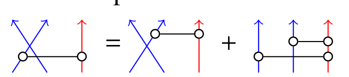
Dots commute with dumbbells:

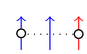
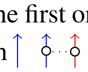
 

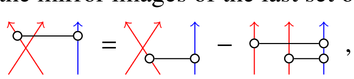
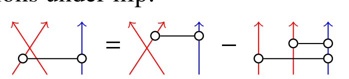
To see this, compose on top and bottom with . Similarly, different dumbbells commute with each other. Also, dumbbells commute past two-color crossings:

For one-color crossings, we have the following more complicated commutation relations:

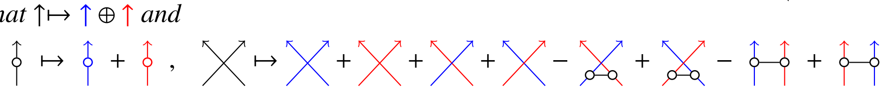
  (4.11)

For example, to prove the first one of these, one just needs to compose on the top with  and on the bottom with , then apply the dot-sliding relation from (4.3). Let us also record the mirror images of the last set of relations under flip:

  (4.12)

We have done this in order to stress that the signs are different when the colors are this way around! Note also in (4.11)–(4.12) that the vertical string on the right hand side could also be drawn on the left hand side; the resulting relations also hold thanks to the commuting relations.

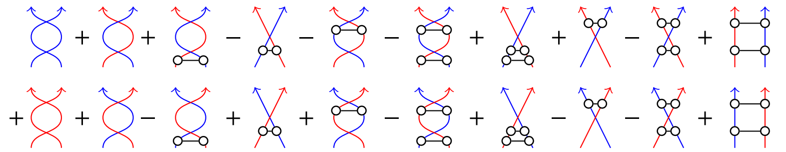
Theorem 4.2. *There is a strict* $\Bbbk$*-linear monoidal functor* $\Delta : \mathcal{AH} \to \mathrm{Add}\left(\mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH}\right)$ *such that* $\uparrow \mapsto \uparrow \oplus \uparrow$ *and*

 (4.13)

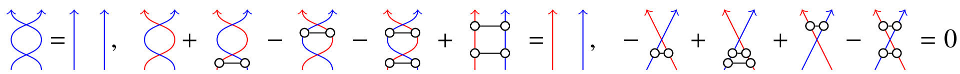
In addition, we have that $\Delta = \mathrm{flip} \circ \Delta$ *(extending* flip *to the additive envelope in the obvious way).*

Remark 4.3. This categorical comultiplication is coassociative like in Remark 3.1.

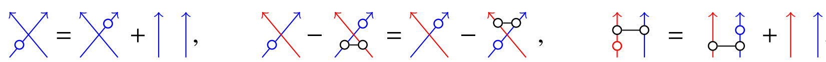
Proof of Theorem 4.2. We just need to check that the defining relations from (3.1) and (4.3) are satisfied in $\mathcal{AH} \,\overline{\odot}\, \mathcal{AH}$. For the quadratic relation, the image of the crossing squared is



This expression is a shorthand for a 4×4 matrix. We must show that it equals the 4×4 identity matrix $\uparrow\uparrow + \uparrow\uparrow + \uparrow\uparrow + \uparrow\uparrow$. Looking at the 16 individual matrix entries (most of which are zero), the proof reduces to verifying the following three identities



together with the mirror images of these identities under flip. All are obviously true by commuting relations. For the dot sliding relation (4.3), one computes the entries of the 4×4 matrices involved to see that the proof reduces to checking the following



together with their mirror images under flip. Again these are all clear; use (4.10) for the last one. The braid relation may be checked by a similar sort of calculation although this is quite lengthy since it involves 8×8 matrices; this is where one needs (4.11)–(4.12). □

The goal now is to show that the new version of the functor Δ also categorifies δ by establishing an analog of (3.12). The canonical functors $\mathcal{AH} \to \mathcal{AH} \,\overline{\odot}\, \mathcal{AH}$ and $\mathcal{AH} \to \mathcal{AH} \,\overline{\odot}\, \mathcal{AH}$ induce a ring homomorphism $\epsilon : K_0(\mathrm{Kar}(\mathcal{AH})) \otimes_{\mathbb{Z}} K_0(\mathrm{Kar}(\mathcal{AH})) \to K_0(\mathrm{Kar}(\mathcal{AH} \,\overline{\odot}\, \mathcal{AH}))$. We claim that

$$\begin{tikzcd} \mathrm{Sym}_{\mathbb{Z}} \arrow[r, "\delta"] \arrow[d, hook, two heads, "\gamma"'] & \mathrm{Sym}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}} \arrow[d, "\epsilon\circ\gamma\otimes\gamma"] \\ K_0(\mathrm{Kar}(\mathcal{AH})) \arrow[r, "{[\Delta]}"'] & K_0(\mathrm{Kar}(\mathcal{AH} \,\overline{\odot}\, \mathcal{AH})) \end{tikzcd} \tag{4.14}$$

commutes. This follows from the next theorem.

Theorem 4.4. *For each $n \geq 0$, we have that*

$$\Delta(H_n) \cong \bigoplus_{r=0}^{n} H_{n-r} \otimes H_r, \qquad\qquad \Delta(E_n) \cong \bigoplus_{r=0}^{n} E_{n-r} \otimes E_r. \tag{4.15}$$

In comparison to Theorem 3.2, the proof of Theorem 4.4 is rather non-trivial, and it will occupy the remainder of the section. We will need the isomorphisms σ_λ ($\lambda \in \mathcal{P}_{r,n}$) from (3.7), viewed now as morphisms in $\mathcal{AH} \,\overline{\odot}\, \mathcal{AH}$. Let us also identify $AH_r \otimes_{\Bbbk} AH_{n-r}$ with a subalgebra of AH_n so that $s_i \otimes 1 \leftrightarrow s_i$, $x_i \otimes 1 \leftrightarrow s_i$, $1 \otimes s_j \leftrightarrow s_{r+j}$ and $1 \otimes x_j \leftrightarrow x_{r+j}$. Let $AH_r \,\overline{\otimes}_{\Bbbk}\, AH_{n-r}$ be the Ore localization of $AH_r \otimes_{\Bbbk} AH_{n-r}$ at the central element

$$z_{r,n} := \prod_{i=1}^{r} \prod_{j=1}^{n-r} (x_i - x_{r+j}). \tag{4.16}$$

Generalizing (3.8), there is an algebra isomorphism

$$\imath_{r,n} : AH_r \,\overline{\otimes}_{\Bbbk}\, AH_{n-r} \to \mathrm{End}_{\mathcal{AH} \,\overline{\odot}\, \mathcal{AH}}(\uparrow^{\otimes(n-r)} \otimes \uparrow^{\otimes r}) \tag{4.17}$$

sending $s_i = s_i \otimes 1$ and $s_{r+j} = 1 \otimes s_j$ to the same diagrams as before, and $x_i = x_i \otimes 1$ and $x_{r+j} = 1 \otimes x_j$ to dots on the ith red string or jth blue string, respectively. To see this, we just observe that the analogous isomorphism before localizing is obvious; then it follows for the localized versions too since all dumbbells make sense in $AH_r \mathbin{\overline{\otimes}}_{\Bbbk} AH_{n-r}$, and conversely the image of $z_{r,n}$ is invertible in the endomorphism algebra. Just like in (3.9), we then get that

$$\mathrm{End}_{\mathrm{Add}(\mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH})}\left((\uparrow \oplus \uparrow)^{\otimes n}\right) \cong \bigoplus_{r=0}^{n} \mathrm{Mat}_{\binom{n}{r}}\left(AH_r \mathbin{\overline{\otimes}}_{\Bbbk} AH_{n-r}\right). \tag{4.18}$$

For $\lambda \in \mathcal{P}_{r,n}$ and $1 \leq i \leq r, 1 \leq j \leq r-n$, we let

$$\varepsilon_{i,j}(\lambda) := \begin{cases} 1 & \text{if } j \leq \lambda_i \\ -1 & \text{if } j > \lambda_i. \end{cases} \tag{4.19}$$

Thus it is 1 or -1 according to whether (i, j) is inside or outside of the Young diagram of λ. Also let

$$y_{i,j} := (x_{r+1-i} - x_{r+j})^{-1} \in AH_r \mathbin{\overline{\otimes}}_{\Bbbk} AH_{n-r}. \tag{4.20}$$

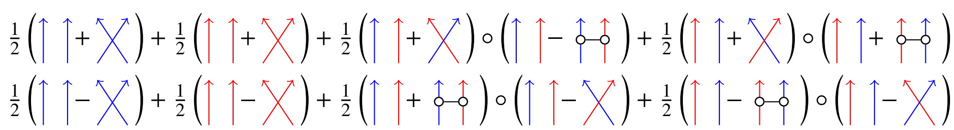
Numbering strings $1, \dots, n$ from right to left as usual, $\imath_{r,n}(y_{i,j})$ is the dumbbell between the $(r+1-i)$th and $(r+j)$th strings; alternatively, numbering strings from the center (with red to the right and blue to the left) it joins the ith red string to the jth blue string. The key observation needed to prove Theorem 4.4 is as follows.

Lemma 4.5. *For $0 \leq r \leq n$ and $\lambda, \mu \in \mathcal{P}_{r,n}$, we have that*

$$1_\mu \circ \Delta(\imath_n(e_{(n)})) \circ 1_\lambda = \binom{n}{r}^{-1} \sigma_\mu^{-1} \circ \imath_{r,n}\left(e_{(r)} \otimes e_{(n-r)}\right) \circ \imath_{r,n}\Big(\prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 + \varepsilon_{i,j}(\lambda) y_{i,j}\right)\Big) \circ \sigma_\lambda,$$

$$1_\mu \circ \Delta(\imath_n(e_{(1^n)})) \circ 1_\lambda = (-1)^{|\lambda|+|\mu|} \binom{n}{r}^{-1} \sigma_\mu^{-1} \circ \imath_{r,n}\Big(\prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 - \varepsilon_{i,j}(\mu) y_{i,j}\right)\Big) \circ \imath_{r,n}\left(e_{(1^r)} \otimes e_{(1^{n-r})}\right) \circ \sigma_\lambda.$$

Proof. Note that $\Delta(\imath_2(e_{(2)}))$ and $\Delta(\imath_2(e_{(1^2)}))$ are equal to



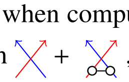
respectively. The lemma in the case $n = 2$ follows from these formulae. For the general case, we proceed by induction on $|\mu| - |\lambda|$. We just explain the proof for the first formula, since the second is similar.

In the base case when $\mu = \min_{r,n}$ (so $1_\mu = \sigma_\mu = \uparrow^{\otimes(n-r)} \otimes \uparrow^{\otimes r}$) and $\lambda = \max_{r,n}$ (so $1_\lambda = \uparrow^{\otimes r} \otimes \uparrow^{\otimes(n-r)}$), we have that

$$e_{(n)} = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_r \times \mathfrak{S}_{n-r}} \sum_{\sigma \in D} \tau\sigma$$

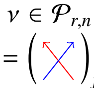
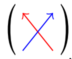
where D denotes the set of minimal length $\mathfrak{S}_r \times \mathfrak{S}_{n-r} \backslash \mathfrak{S}_n$-coset representatives. For $\tau \in \mathfrak{S}_r \times \mathfrak{S}_{n-r}$, we have that $1_\mu \circ \Delta(\imath_n(\tau)) = \sigma_\mu^{-1} \circ \imath_{r,n}(\tau) \circ 1_\mu$. Thus, we see that

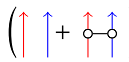
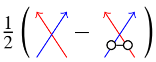
$$1_\mu \circ \Delta(\imath_n(e_{(n)})) \circ 1_\lambda = \binom{n}{r}^{-1} \sum_{\sigma \in D} \sigma_\mu^{-1} \circ \imath_{r,n}(e_{(r)} \otimes e_{(n-r)}) \circ 1_\mu \circ \Delta(\imath_n(\sigma)) \circ 1_\lambda.$$

Since λ is maximal, the term $1_\mu \circ \Delta(\imath_n(\sigma)) \circ 1_\lambda$ here can only be non-zero when σ is the longest coset representative. Moreover, when computing $\Delta(\imath_n(\sigma))$, we must replace each crossing  in a reduced word for $\imath_n(\sigma)$ with , i.e., the terms from the expression in (4.13) that are colored $\uparrow\uparrow$ at the top and $\uparrow\uparrow$ at the bottom. We conclude for this longest σ that

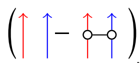
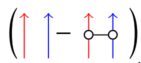
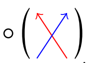
$$1_\mu \circ \Delta(\imath_n(\sigma)) \circ 1_\lambda = \imath_{r,n}\Big(\prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 + y_{i,j}\right)\Big) \circ \sigma_\lambda.$$

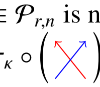
Since $\varepsilon_{i,j}(\lambda) = 1$ for all i and j, this checks the base case.

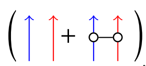
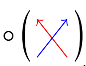
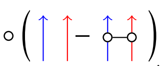
For the induction step, take $\mu, \lambda \in \mathcal{P}_{r,n}$ such that either μ is not minimal or λ is not maximal, and consider $X := 1_\mu \circ \Delta(\iota_n(e_{(n)})) \circ 1_\lambda$. If μ is not minimal, we let $\nu \in \mathcal{P}_{r,n}$ be obtained from μ by removing a box. Let j be the unique index such that that $\sigma_\mu^{-1} =$  $_j \circ \sigma_\nu^{-1}$, where the subscript indicates we are applying the crossing to the jth and $(j+1)$th strings. The induction hypothesis gives us a formula for $Y := 1_\nu \circ \Delta(\iota_n(e_{(n)})) \circ 1_\lambda$, reducing the problem to showing that $X =$  $_j \circ Y$. To see this, we apply $1_\mu \circ \Delta(\iota_n(-)) \circ 1_\lambda$ to the identity $e_{(n)} = \frac{1}{2}(1 + s_j)e_{(n)}$ to deduce that

$X = \frac{1}{2}$  $_j \circ X + \frac{1}{2}$  $_j \circ Y.$

Hence,

 $_j \circ X =$  $_j \circ$  $_j \circ Y.$

In view of the isomorphism (4.18), this morphism space is free as a module over the integral domain $\Bbbk[x_1, \dots, x_n]_{z_{r,n}}$, so it is permissible to cancel the first term, and this gives the desired formula. Instead, if $\lambda \in \mathcal{P}_{r,n}$ is not maximal, we let κ be obtained from λ by adding a box, and define j so that $\sigma_\lambda = \sigma_\kappa \circ$  $_j$. Let $Z := 1_\mu \circ \Delta(\iota_n(e_{(n)})) \circ 1_\kappa$. Then we need to show that

$X \circ$  $_j = Z \circ$  $_j \circ$  $_j,$

which follows by applying $1_\mu \circ \Delta(\iota_n(-)) \circ 1_\lambda$ to the identity $e_{(n)} = e_{(n)}\frac{1}{2}(1 + s_j)$. $\square$

From (4.1), one sees that $s_i f e_{(n)} = (s_i \oplus f)e_{(n)}$ and $s_i f e_{(1^n)} = -(s_i \ominus f)e_{(1^n)}$, where

$$s_i \oplus f := s_i(f) + \partial_i(f), \qquad s_i \ominus f := s_i(f) - \partial_i(f). \tag{4.21}$$

Transporting the left action of AH_n on $AH_n e_{(n)}$ through the linear isomorphism $\Bbbk[x_1, \dots, x_n] \xrightarrow{\sim} AH_n e_{(n)}, f \mapsto f e_{(n)}$, we deduce that $\Bbbk[x_1, \dots, x_n]$ is a left AH_n-module with $\Bbbk[x_1, \dots, x_n]$ acting by left multiplication and $\mathfrak{S}_n$ acting by $\oplus$. By degree considerations, the space of $\mathfrak{S}_n$-fixed points with respect to the action $\oplus$ is the same as the fixed points with respect to the usual action, i.e., we recover the subalgebra Sym_n of $\Bbbk[x_1, \dots, x_n]$. This shows that the *spherical subalgebra* $e_{(n)}AH_n e_{(n)}$ of AH_n is Sym_n. Moreover, for any $f \in \Bbbk[x_1, \dots, x_n]$, we have that

$$e_{(n)} f e_{(n)} = \frac{1}{n!}\sum_{\pi \in \mathfrak{S}_n} e_{(n)} \pi f e_{(n)} = e_{(n)}\left(\frac{1}{n!}\sum_{\pi \in \mathfrak{S}_n} \pi \oplus f\right) e_{(n)}. \tag{4.22}$$

Similarly, one sees that $e_{(1^n)}AH_n e_{(1^n)} = \mathrm{Sym}_n$ and

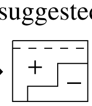
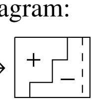
$$e_{(1^n)} f e_{(1^n)} = e_{(1^n)}\left(\frac{1}{n!}\sum_{\pi \in \mathfrak{S}_n} \pi \ominus f\right) e_{(1^n)}. \tag{4.23}$$

The $\oplus$ and $\ominus$ actions extend to actions on $\Bbbk(x_1, \dots, x_n)$, with the simple transpositions satisfying the same formulae (4.21).

Lemma 4.6. *For $0 \leq r \leq n$, we have that*

$$\sum_{\pi \in \mathfrak{S}_r \times \mathfrak{S}_{n-r}} \pi \oplus \Big(\sum_{\lambda \in \mathcal{P}_{r,n}} \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 + \varepsilon_{i,j}(\lambda) y_{i,j}\right)\Big) = n! = \sum_{\pi \in \mathfrak{S}_r \times \mathfrak{S}_{n-r}} \pi \ominus \Big(\sum_{\mu \in \mathcal{P}_{r,n}} \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 - \varepsilon_{i,j}(\mu) y_{i,j}\right)\Big).$$

Proof. We just explain the proof of the first equality; the second then follows by considering the automorphism $x_i \mapsto -x_i$ of $\Bbbk(x_1, \dots, x_n)$. Proceed by induction on n. For the induction step, we partition $\mathcal{P}_{r,n}$ as $A \sqcup B$ as suggested by the diagram:

$A \leftrightarrow$  , $\qquad B \leftrightarrow$  .

Thus, A consists of $\lambda \in \mathcal{P}_{r,n}$ such that $\lambda_1 = n - r$, and B consists of $\lambda \in \mathcal{P}_{r,n}$ such that $\lambda_1 < n - r$. The expression we are trying to compute then splits as a sum $X + Y$ where for X we take the second sum just over $\lambda \in A$ and for Y we take it over $\lambda \in B$. Using the induction hypothesis plus the observation that $\{1, s_{n-1}, s_{n-2}s_{n-1}, \dots, s_1, \dots, s_{n-1}\}$ is a set of $\mathfrak{S}_n/\mathfrak{S}_{n-1}$-coset representatives, we see that

$$X = (n-1)!(1 + s_{r-1} + \cdots + s_1 \cdots s_{r-1}) \oplus \prod_{j=1}^{n-r} \left(1 + (x_r - x_{r+j})^{-1}\right),$$

$$Y = (n-1)!(1 + s_{n-1} + \cdots + s_{r+1} \cdots s_{n-1}) \oplus \prod_{i=1}^{r} \left(1 - (x_i - x_n)^{-1}\right).$$

It remains to show that $X+Y = n!$. The constant terms of X and Y are $r(n-1)!$ and $(n-r)(n-1)!$, which sum to $n!$. So we need to show that the remaining terms X_+ and Y_+ of X and Y satisfy $X_+ + Y_+ = 0$.

To do this, we work in the larger algebra $\Bbbk[\![x_1^{-1}, \dots, x_r^{-1}, x_{r+1}, \dots, x_n]\!]$ of formal power series. In this algebra, we have that

$$\prod_{j=1}^{n-r} \left(1 + (x_r - x_{r+j})^{-1}\right) = \prod_{j=1}^{n-r} \left(1 + x_r^{-1} + x_r^{-2}x_{r+j} + \cdots\right) = 1 + \sum_{m \geq 0} \sum_{\substack{q_1,\dots,q_{n-r} \geq 0 \\ q_1+\cdots+q_{n-r}=m+1}} x_r^{-m-1} \prod_{\substack{1 \leq j \leq n-r \\ q_j > 0}} x_{r+j}^{q_j-1}$$

$$\prod_{i=1}^{r} \left(1 - (x_i - x_n)^{-1}\right) = \prod_{i=1}^{r} \left(1 - x_i^{-1} - x_i^{-2}x_n - \cdots\right) = \sum_{m \geq 0} \sum_{\substack{p_1,\dots,p_r \geq 0 \\ p_1+\delta_{p_1,0}+\cdots+p_r+\delta_{p_r,0}=m+r}} x_n^m \prod_{\substack{1 \leq i \leq r \\ p_i > 0}} \left(-x_i^{-p_i}\right).$$

A computation using the definitions (4.2) and (4.21) gives for $m \geq 0$ that

$$(1 + s_{r-1} + \cdots + s_1 \cdots s_{r-1}) \oplus x_r^{-m-1} = \delta_{m,0} - \sum_{\substack{p_1,\dots,p_r \geq 0 \\ p_1+\delta_{p_1,0}+\cdots+p_r+\delta_{p_r,0}=m+r}} \prod_{\substack{1 \leq i \leq r \\ p_i > 0}} \left(-x_i^{-p_i}\right),$$

$$(1 + s_{n-1} + \cdots + s_{r+1} \cdots s_{n-1}) \oplus x_n^m = \sum_{\substack{q_1,\dots,q_{n-r} \geq 0 \\ q_1+\cdots+q_{n-r}=m+1}} \prod_{\substack{1 \leq j \leq n-r \\ q_j > 0}} x_{r+j}^{q_j-1}.$$

Putting these things together, we deduce that

$$-X_+ = Y_+ = (n-1)! \sum_{m \geq 0} \left(\sum_{\substack{p_1,\dots,p_r \geq 0 \\ p_1+\delta_{p_1,0}+\cdots+p_r+\delta_{p_r,0}=m+r}} \prod_{\substack{1 \leq i \leq r \\ p_i > 0}} \left(-x_i^{-p_i}\right) \right) \left(\sum_{\substack{q_1,\dots,q_{n-r} \geq 0 \\ q_1+\cdots+q_{n-r}=m+1}} \prod_{\substack{1 \leq j \leq n-r \\ q_j > 0}} x_{r+j}^{q_j-1} \right).$$

Hence, $X_+ + Y_+ = 0$. □

For later reference, let us also discuss the space $e_{(1^n)}AH_n e_{(n)}$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$, let $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and $A_\lambda := \sum_{\pi \in \mathfrak{S}_n} (-1)^{\ell(\pi)} \pi(x^\lambda)$. Setting $\rho := (n-1, \dots, 1, 0) \in \mathbb{N}^n$, the symmetric polynomial

$$\mathcal{X}_\lambda := A_{\lambda+\rho}/A_\rho \in \mathrm{Sym}_n \tag{4.24}$$

is the usual Schur polynomial in n variables when $\lambda_1 \geq \cdots \geq \lambda_n$; on the other hand, it is zero if $\lambda + \rho$ has a repeated entry. We have that $e_{(1^n)}(\ker \partial_i)e_{(n)} = 0$, hence, $e_{(1^n)}s_i(f)e_{(n)} = -e_{(1^n)}fe_{(n)}$. Since $\Bbbk[x_1, \dots, x_n] = (\ker \partial_1 + \cdots + \ker \partial_{n-1}) \oplus \mathrm{Sym}_n\, x^\rho$, we deduce that $e_{(1^n)}AH_n e_{(n)}$ is a free Sym_n-module generated by $e_{(1^n)}x^\rho e_{(n)}$. Moreover,

$$e_{(1^n)}x^\lambda e_{(n)} = \mathcal{X}_{\lambda-\rho} e_{(1^n)}x^\rho e_{(n)} = e_{(1^n)}x^\rho e_{(n)} \mathcal{X}_{\lambda-\rho} \tag{4.25}$$

for any $\lambda \in \mathbb{N}^n$. Similar statements hold when $e_{(n)}$ and $e_{(1^n)}$ are interchanged.

Proof of Theorem 4.4. Consider first the statement about H_n. Exactly like in the proof of Theorem 3.2, we need to construct morphisms u and v in $\mathrm{Kar}(\mathcal{AH} \mathbin{\overline{\odot}} \mathcal{AH})$ such that $u \circ v = \Delta(\iota_n(e_{(n)}))$ and $v \circ u = \sum_{r=0}^{n} \iota_{r,n}(e_{(r)} \otimes e_{(n-r)})$. We set

$$u := \sum_{r=0}^{n} \binom{n}{r}^{-1} \sum_{\mu \in \mathcal{P}_{r,n}} \sigma_\mu^{-1} \circ \iota_{r,n}(e_{(r)} \otimes e_{(n-r)}),$$

$$v := \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_{r,n}} \iota_{r,n}\left(e_{(r)} \otimes e_{(n-r)}\right) \circ \iota_{r,n}\Bigg(\prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 + \varepsilon_{i,j}(\lambda) y_{i,j}\right)\Bigg) \circ \sigma_\lambda.$$

Lemma 4.5 implies that $u \circ v = \Delta(\iota_n(e_{(n)}))$. Also

$$v \circ u = \sum_{r=0}^{n} \binom{n}{r}^{-1} \iota_{r,n}\left(e_{(r)} \otimes e_{(n-r)}\right) \circ \iota_{r,n}\Bigg(\sum_{\lambda \in \mathcal{P}_{r,n}} \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 + \varepsilon_{i,j}(\lambda) y_{i,j}\right)\Bigg) \circ \iota_{r,n}\left(e_{(r)} \otimes e_{(n-r)}\right).$$

Using the analog of (4.22) for $AH_r \otimes AH_{n-r}$, this equals

$$\sum_{r=0}^{n} \iota_{r,n}\left(e_{(r)} \otimes e_{(n-r)}\right) \circ \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_r \times \mathfrak{S}_{n-r}} \pi \oplus \Bigg(\sum_{\lambda \in \mathcal{P}_{r,n}} \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 + \varepsilon_{i,j}(\lambda) y_{i,j}\right)\Bigg) \circ \iota_{r,n}\left(e_{(r)} \otimes e_{(n-r)}\right).$$

Then we use Lemma 4.6 to see that this equals the required $\sum_{r=0}^{n} \iota_{r,n}(e_{(r)} \otimes e_{(n-r)})$.

For the statement about E_n, we need morphisms u and v such that $u \circ v = \Delta(\iota_n(e_{(1^n)}))$ and $v \circ u = \sum_{r=0}^{n} \iota_{r,n}(e_{(1^r)} \otimes e_{(1^{n-r})})$. One takes

$$u := \sum_{r=0}^{n} \binom{n}{r}^{-1} \sum_{\mu \in \mathcal{P}_{r,n}} (-1)^{|\mu|} \sigma_\mu^{-1} \circ \iota_{r,n}\Bigg(\prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}} \left(1 - \varepsilon_{i,j}(\mu) y_{i,j}\right)\Bigg) \circ \iota_{r,n}(e_{(1^r)} \otimes e_{(1^{n-r})}),$$

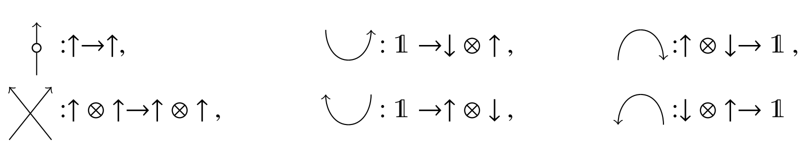
$$v := \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_{r,n}} (-1)^{|\lambda|} \iota_{r,n}\left(e_{(1^r)} \otimes e_{(1^{n-r})}\right) \circ \sigma_\lambda.$$

The proof then proceeds like in the previous paragraph, using (4.23) instead of (4.22). $\square$

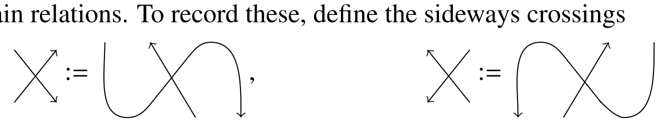
5. The degenerate Heisenberg category

Although for us $\Bbbk$ is a field of characteristic zero, the following definition makes sense for $\Bbbk$ that is any commutative ring. Moreover, all of the results recorded in this section are valid for any $\Bbbk$, including the definition of the categorical comultiplication in Theorem 5.3 (but excluding (5.36) since $n!$ needs to be invertible for the underlying idempotents to be defined).

Definition 5.1 ([B, Theorem 1.2])**.** The *(degenerate) Heisenberg category* $\mathcal{H}eis_k$ of central charge $k \in \mathbb{Z}$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow$ and $\downarrow$ and morphisms

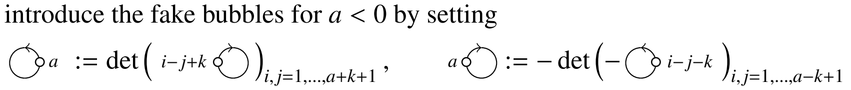


subject to certain relations. To record these, define the sideways crossings



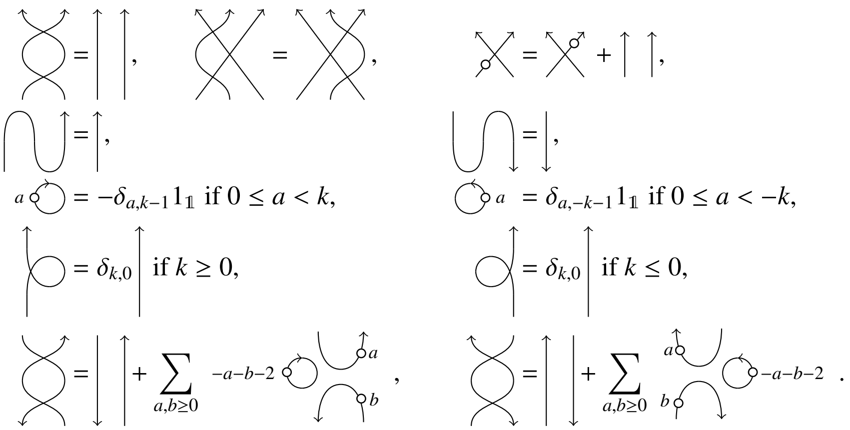
(5.1)

and introduce the fake bubbles for $a < 0$ by setting



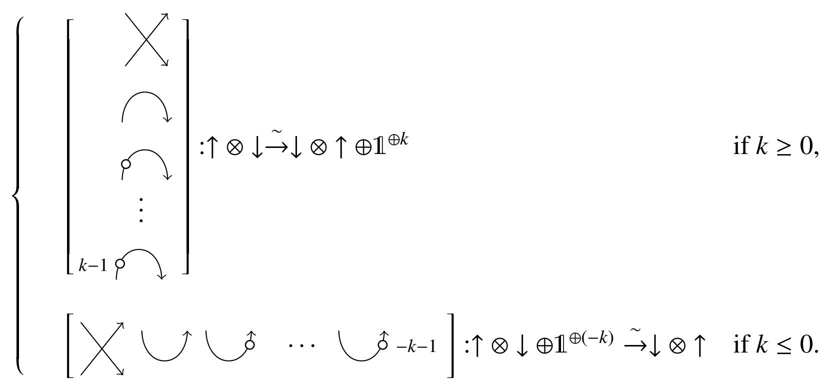
(5.2)

interpreting the determinant of an $n \times n$ matrix as $\delta_{n,0}$ in case $n \leq 0$. Then the relations are as follows:



(5.3)
(5.4)
(5.5)
(5.6)
(5.7)

As explained in the proof of [B, Theorem 1.2], the defining relations of $\mathcal{H}eis_k$ imply that the following is an isomorphism in Add($\mathcal{H}eis_k$):



(5.8)

In fact, as in [B, Definition 1.1], $\mathcal{H}eis_k$ can be defined equivalently as the strict $\Bbbk$-linear monoidal category generated by the morphisms , , and subject just to the relations (5.3)–(5.4) plus the requirement that the morphism (5.8) is invertible (where the rightward crossing is defined as in (5.1)). In the category defined in this way, there are then unique morphisms and such that the other relations (5.5)–(5.7) from Definition 5.1 hold.

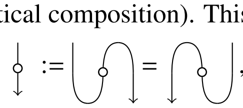
We will need various other relations in $\mathcal{H}eis_k$, most of which are derived in [B, Theorem 1.3]. The relation (5.4) means that $\downarrow$ is a right dual to $\uparrow$. It is also a left dual since the following relations hold:

(5.9)

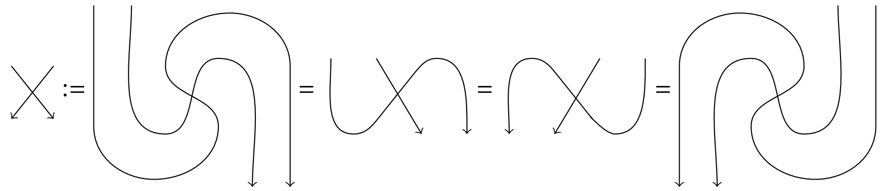
This means that $\mathcal{H}eis_k$ is *rigid*. Moreover, it is *strictly pivotal*: rotating diagrams through 180° defines a strict $\Bbbk$-linear monoidal isomorphism

$$* : \mathcal{H}eis_k \to ((\mathcal{H}eis_k)^{\mathrm{op}})^{\mathrm{rev}}, \tag{5.10}$$

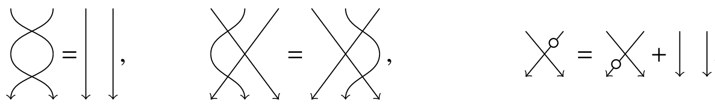
where op (resp., rev) denotes the monoidal category with the same horizontal composition and the opposite vertical composition (resp., the reversed horizontal composition and the same vertical composition). This follows due to the relations



(5.11)

 (5.12)

Informally, these relations mean that dots and crossings slide over cups and caps. Applying $*$ to the relations (5.3) and (5.6) gives

 (5.13)

 (5.14)

There is another useful symmetry

$$\Omega_k : \mathcal{H}eis_k \xrightarrow{\sim} (\mathcal{H}eis_{-k})^{\mathrm{op}}\,, \tag{5.15}$$

which sends a morphism in $\mathcal{H}eis_k$ represented by some string diagram to the morphism in $\mathcal{H}eis_{-k}$ obtained by reflecting this diagram in a horizontal axis then multiplying by $(-1)^{x+y}$, where x is the total number of crossings and y is the total number of leftward cups and caps in the diagram (including ones in fake bubbles); see [B, Lemma 2.1].

Remark 5.2. Using $*$ and Ω_k, one can deduce several more equivalent presentations for $\mathcal{H}eis_k$. For example, it may be defined by the same generating objects and morphisms as in Definition 5.1 subject to the relations (5.3), (5.9), (5.5), (5.6) and (5.7); i.e., we have traded the right adjunction relation (5.4) for the left adjunction relation (5.9). Alternatively, one could replace the generating morphisms given by the upward dot and crossing with the downward dot and crossing, taking the relations (5.13), (5.9), (5.5), (5.14) and (5.7), where the sideways crossings are obtained by rotating the downward one in an analogous way to (5.1). There are also alternative versions of both of these presentations based on an "inversion relation" along the lines of the presentation explained after (5.8).

Since the relations (3.1) and (4.3) hold in $\mathcal{H}eis_k$, there is a strict $\Bbbk$-linear monoidal functor $\imath : \mathcal{AH} \to \mathcal{H}eis_k$ sending diagrams in $\mathcal{AH}$ to the same diagrams viewed instead as morphisms in $\mathcal{H}eis_k$; this is actually an inclusion thanks to the basis theorem established in Theorem 6.4 below, but we will not use this fact here. In particular, this means that there is an algebra homomorphism

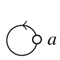
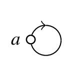
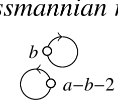
$$\imath_n : AH_n \to \mathrm{End}_{\mathcal{H}eis_k}(\uparrow^{\otimes n}) \tag{5.16}$$

sending s_i to the crossing of the ith and $(i+1)$th strings, and x_j to the dot on the jth string. Using (5.13), one sees also that there is an algebra homomorphism

$$\jmath_n : AH_n \to \mathrm{End}_{\mathcal{H}eis_k}(\downarrow^{\otimes n}) \tag{5.17}$$

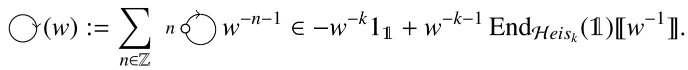
sending $-s_i$ to the crossing of the ith and $(i+1)$th strings, and x_j to the dot on the jth string. Note $\imath_n$ and $\jmath_n$ are related by the formula $\jmath_n = \Omega_k \circ \imath_n \circ \tau$ where $\tau : AH_n \to AH_n$ is the antiautomorphism which is the identity on each of the generators s_i and x_j.

The bubbles (both genuine and fake) satisfy the *infinite Grassmannian relations*:

 $= \delta_{a,-k-1}1_{\mathbb{1}}$ if $a < -k$,  $= -\delta_{a,k-1}1_{\mathbb{1}}$ if $a < k$, $\sum_{b\in\mathbb{Z}}$  $= -\delta_{a,0}1_{\mathbb{1}}$, (5.18)

for any $a \in \mathbb{Z}$. For an indeterminate w, let

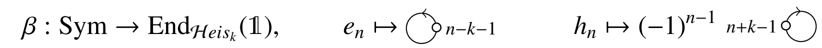
 $(w) := \sum_{n\in\mathbb{Z}}$  $w^{-n-1} \in w^k 1_{\mathbb{1}} + w^{k-1}\,\mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})[\![w^{-1}]\!]$, (5.19)

 (5.20)

Then the infinite Grassmannian relation implies that

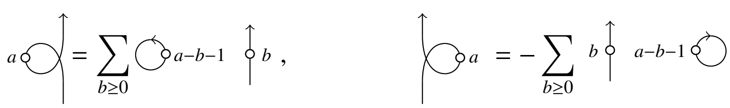
 (5.21)

Up to the choice of normalization, this is the well-known identity from [M, (I.2.6)] relating elementary and complete symmetric functions. It follows that there is a well-defined algebra homomorphism

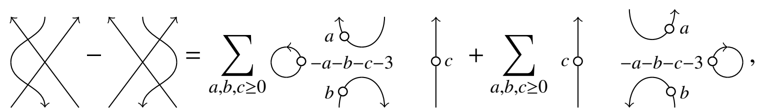
 (5.22)

where $\mathrm{Sym} := \Bbbk \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}$ denotes the algebra of symmetric functions over $\Bbbk$. Using this dictionary, one can also make sense of the determinantal formulae (5.2) used to define the fake bubbles in Definition 5.1: they are are a formal consequence of another well-known symmetric functions identity from [M, Exercise I.2.8].

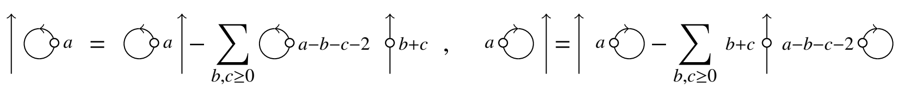
There are three more essential relations: the *curl relations*

 (5.23)

for all $a \geq 0$, the *alternating braid relation*

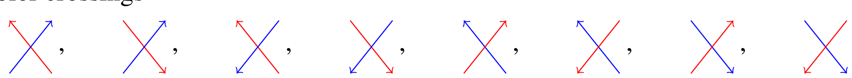
 (5.24)

and the *bubble slides*

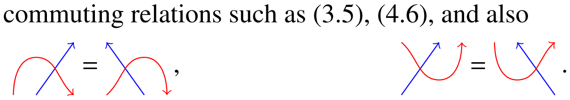
 (5.25)

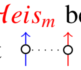
for $a \in \mathbb{Z}$.

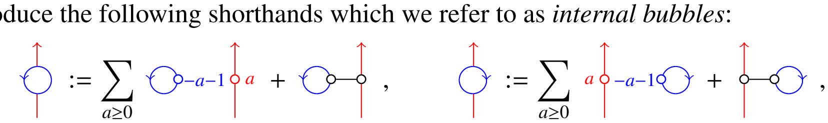
For the remainder of the section, we assume that $k = l + m$ for integers $l, m \in \mathbb{Z}$. Let $\mathcal{H}eis_l \odot \mathcal{H}eis_m$ be the symmetric product defined as in Section 3. Now there are additional two-color crossings

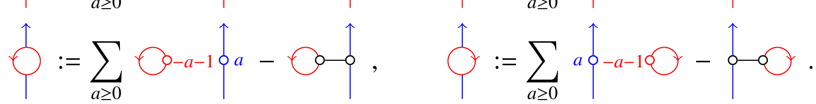


These satisfy many commuting relations such as (3.5), (4.6), and also

 (5.26)

We extend the notation (4.7) to $\mathcal{H}eis_l \odot \mathcal{H}eis_m$ in the obvious way. Let $\mathcal{H}eis_l \mathbin{\bar{\odot}} \mathcal{H}eis_m$ be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{H}eis_l \odot \mathcal{H}eis_m$ by localizing at . As in (4.8), we denote the two-sided inverse of this morphism by a solid dumbbell. Then we introduce the following shorthands which we refer to as *internal bubbles*:

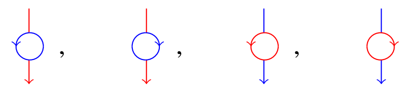
 (5.27)

 (5.28)

The category $\mathcal{H}eis_l \mathbin{\bar{\odot}} \mathcal{H}eis_m$ is strictly pivotal with duality functor

$$* : \mathcal{H}eis_l \mathbin{\bar{\odot}} \mathcal{H}eis_m \to \left(\left(\mathcal{H}eis_l \mathbin{\bar{\odot}} \mathcal{H}eis_m\right)^{\mathrm{op}}\right)^{\mathrm{rev}} \tag{5.29}$$

defined by rotating diagrams through 180°. In particular, the left mates of the internal bubbles (5.27)–(5.28) are equal to their right mates. We denote these by

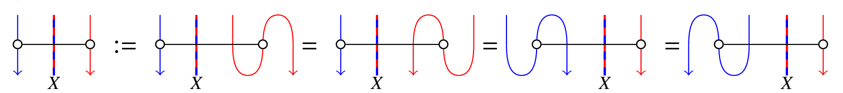


This definition ensures that internal bubbles commute past cups and caps in all possible configurations. For example:

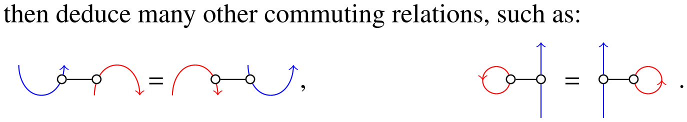
The dotted line notation (4.7) obviously makes sense between points on downward strings as well as on upward strings. We extend the dumbbell notation to such situations by setting

These are two-sided inverses of the morphisms represented by the same diagrams but with dotted lines replacing the solid ones in the dumbbells. The morphisms on the right hand sides of the following are also such inverses, hence, these are true equations due to the uniqueness of inverses:

Similarly,



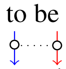
From this discussion, it follows that dumbbells commute over cups and caps in any configuration. One can then deduce many other commuting relations, such as:



We will appeal to these sorts of relation without further mention. Finally, there are two more useful symmetries

$$\mathrm{flip} : \mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m \xrightarrow{\sim} \mathcal{H}eis_m \mathbin{\overline{\odot}} \mathcal{H}eis_l, \tag{5.30}$$

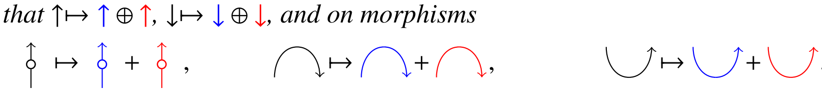
$$\Omega_{l|m} : \mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m \xrightarrow{\sim} \left(\mathcal{H}eis_{-l} \mathbin{\overline{\odot}} \mathcal{H}eis_{-m}\right)^{\mathrm{op}}. \tag{5.31}$$

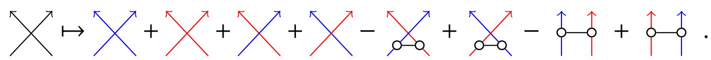
The first of these is defined on diagrams by switching the colors blue and red then multiplying by $(-1)^z$ where z is the total number of dumbbells in the picture; it interchanges the internal bubbles in (5.27) with the ones in (5.28). The second takes a diagram to its mirror image in a horizontal axis multiplied by $(-1)^{x+y}$ where x is the number of one-colored crossings and y is the number of leftward cups and caps (including ones in fake and internal bubbles). The only additional thing that needs to be used to see that this is well defined beyond what was already checked for (5.15) is that  is invertible. All of the symmetries $*$, flip and $\Omega_{l|m}$ extend canonically to the Karoubi envelope.

Theorem 5.3. *For $k = l + m$ as above, there is a unique strict $\Bbbk$-linear monoidal functor*

$$\Delta_{l|m} : \mathcal{H}eis_k \to \mathrm{Add}\left(\mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m\right)$$

such that $\uparrow \mapsto \uparrow \oplus \uparrow$, $\downarrow \mapsto \downarrow \oplus \downarrow$, and on morphisms

 (5.32)

 (5.33)

We have that flip $\circ\Delta_{l|m} = \Delta_{m|l}$. *Moreover,* $\Delta_{l|m}$ *satisfies the following for all* $a \in \mathbb{Z}$*:*

 (5.34)

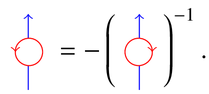
 (5.35)

Finally, extending $\Delta_{l|m}$ *to the Karoubi envelopes in the canonical way, we have that*

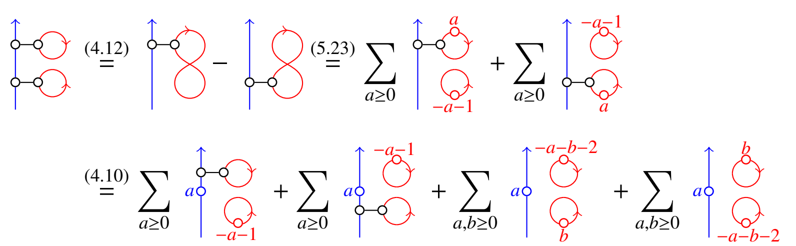
$$\Delta_{l|m}(H_n^\pm) \cong \bigoplus_{r=0}^{n} H^\pm_{n-r} \otimes H^\pm_r, \qquad \Delta_{l|m}(E_n^\pm) \cong \bigoplus_{r=0}^{n} E^\pm_{n-r} \otimes E^\pm_r. \tag{5.36}$$

Remark 5.4. As in Remarks 3.1 and 4.3, the categorical comultiplication $\Delta_{l|m}$ is coassociative in the appropriate sense. It also seems worth pointing out that $\Delta_{l|m}$ does *not* commute either with the duality $*$ or the involution Ω. In fact, either of the monoidal functors $* \circ \Delta_{l|m} \circ *$ or $\Omega_{-l|-m} \circ \Delta_{-l|-m} \circ \Omega_k$ could be used as different (but equally natural) choices for the categorical comultiplication map.

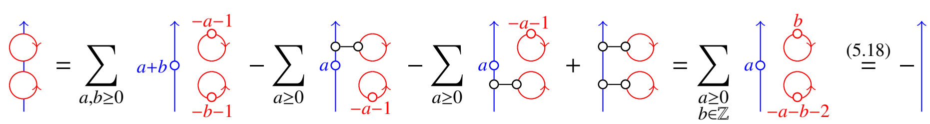
The proof of Theorem 5.3 will be explained at the end of the section. The main work is to verify that all of the defining relations from Definition 5.1 are satisfied in $\mathcal{H}eis_l \,\overline{\odot}\, \mathcal{H}eis_m$. To prepare for this, we first establish a series of lemmas.

Lemma 5.5. *We have that* 

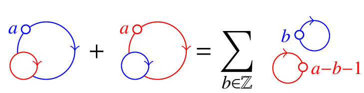
Proof. We note first that



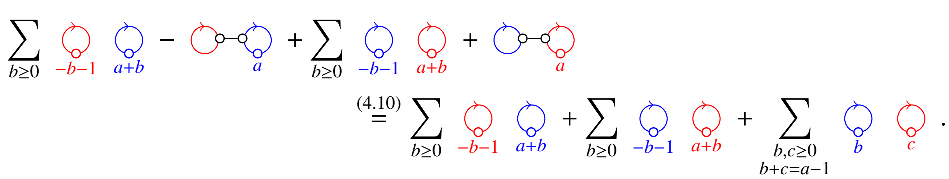
Using this and the definition (5.28) gives that



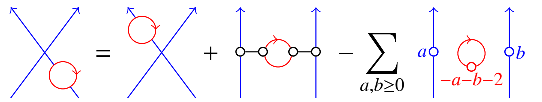
Noting that internal bubbles on the same string commute, this implies the result. ◻

Lemma 5.6. *We have that*  *for any* $a \geq 0$.

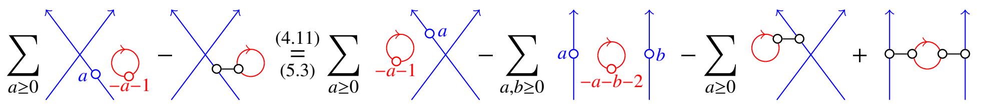
Proof. By the definitions (5.27)–(5.28), the left hand side is



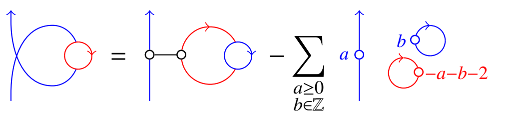
This simplifies to produce the single summation on the right hand side. ◻

Lemma 5.7. *We have that* 

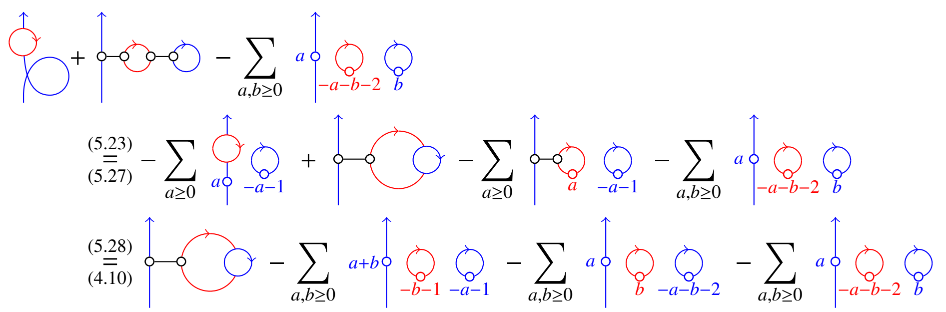
Proof. By the definition (5.28), the left hand side equals



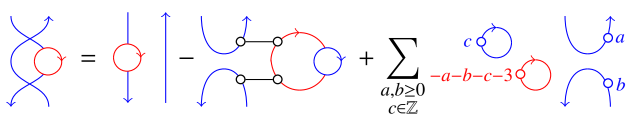
Using (5.28) once again, this is equal to the right hand side. □

Lemma 5.8. *We have that* 

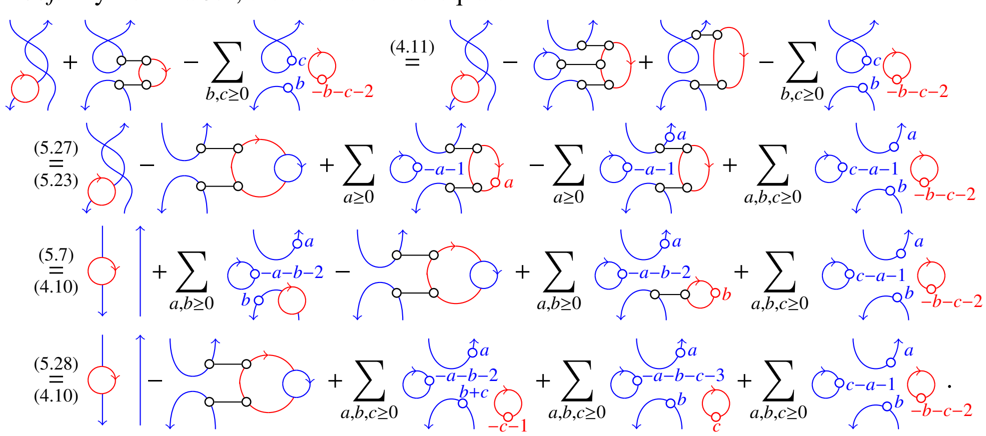
Proof. We apply Lemma 5.7 to commute the internal bubble in the term on the left hand side past the crossing. The left hand side becomes



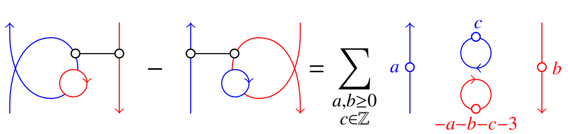
which equals the right hand side. □

Lemma 5.9. *We have that* 

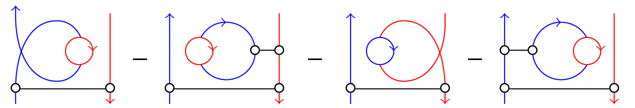
Proof. By Lemma 5.7, the left hand side equals



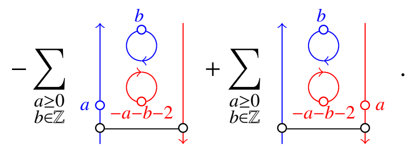
It remains to observe that the three summations at the end simplify to the single summation on the right hand side of the formula we are trying to prove. □

Lemma 5.10. *We have that* 

Proof. Applying (4.11)–(4.12), the left hand side equals

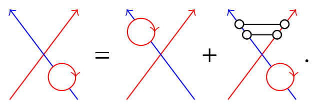


Then we apply Lemma 5.7 (resp., the relation obtained by applying flip to its 180° rotation) to commute the internal bubble in the first (resp., third) term past the crossing. The second and fourth terms cancel terms in the result, and our expression becomes

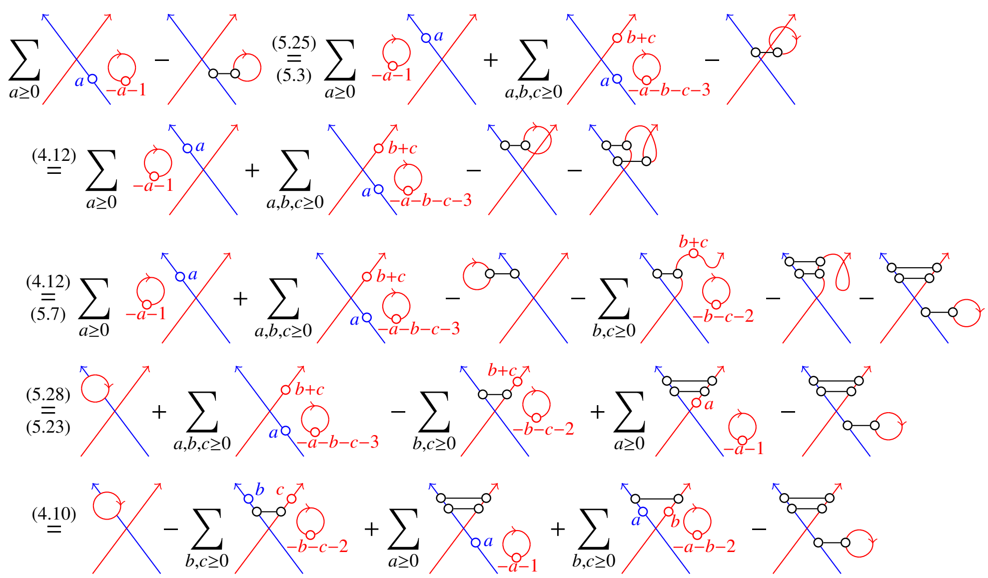


Now an application of (4.10) gives the result. □

Lemma 5.11. *We have that*



Proof. By (5.28), the left hand side is

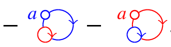
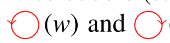


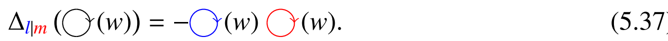
which is equal to the right hand side by (5.28) once again. □

Proof of Theorem 5.3. Once Δ has been constructed, the part about flip is obvious. Also (5.36) for the sign + follows from Theorem 4.4, noting that the formulae in (4.13) are the same as here; then for the sign − it follows by taking right duals (using the rightward cups and caps).

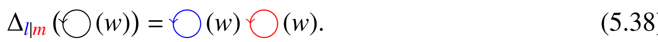
In the remainder of the proof, we are going to use the presentation from Definition 5.1 to establish the existence of $\Delta_{l|m}$. Thus we must check that the images of the defining relations (5.3)–(5.7) under the functor $\Delta_{l|m}$ all hold in Add($\mathcal{H}eis_l \mathbin{\overline{\odot}} \mathcal{H}eis_m$). Moreover we will show that $\Delta_{l|m}$ satisfies (5.35). This is enough to prove the theorem as stated since, in the presence of the relations (5.3)–(5.7), the leftward cup and cap are uniquely determined by the other generators. We already checked the relations (5.3) in the proof of Theorem 4.2. Also the check of the relation (5.4) is quite trivial since all of the matrices involved are diagonal.

Next we check (5.35). Assume that $k \geq 0$ and consider the clockwise bubble a◯. If it is a fake bubble, i.e., $a < 0$, it is a scalar (usually zero) by the definition (5.2) and the assumption on k. Hence, it is quite trivial to see that (5.35) is satisfied. When $a \geq 0$, the image of a◯ under

$\Delta_{l|m}$ is $-$ , which is indeed equal to $-\sum_{b\in\mathbb{Z}}$  by Lemma 5.6. Now consider (5.35) for the counterclockwise bubble (still assuming $k \geq 0$). Define the generating functions  (resp., ) in the same way as (5.19)–(5.20) but using blue (resp., red) bubbles in place of black ones. We have proved already that



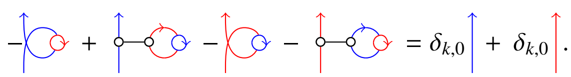
Passing to the inverses of these formal power series and using (5.21) shows that



Equating coefficients yields the desired relation for the counterclockwise bubble. This completes the proof of (5.35) when $k \geq 0$. A similar argument works when $k \leq 0$ too: one starts off by considering the relation for the counterclockwise bubble, using the infinite Grassmannian relation to deduce the one for the clockwise bubble at the end. On the way, one needs to use the relation obtained by applying the symmetry $\Omega_{l|m}$ to Lemma 5.6.

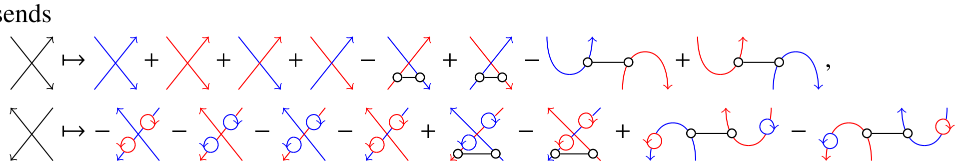
The relation (5.5) follows easily from (5.35) using the first two equalities in (5.18) for the blue and red bubbles.

Moving on to (5.6), we first consider the right curl, so $k \geq 0$. Applying $\Delta_{l|m}$ to the relation reveals that we must show

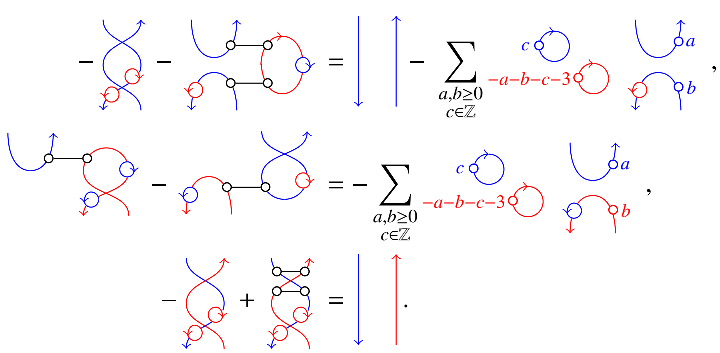


This follows from the identity in Lemma 5.8 and its mirror image under flip. Note for this that the only non-zero term in the summation on the right hand side of this identity is the one with $a = b = 0$ due to the assumption that $k \geq 0$. The argument to treat the case of the left curl is entirely similiar; it depends ultimately on the identity obtained by applying the symmetry $\Omega_{l|m}$ to Lemma 5.8 then rotating through 180°.

Finally, we must check (5.7). We just go through the argument for this for the first equation. The proof of the second one is entirely similar; it depends ultimately on three identities derived from Lemmas 5.9–5.11 by applying $\Omega_{l|m}$ then rotating. By the definition (5.1), the map $\Delta_{l|m}$ sends



With this, it is straightforward to compute the image under $\Delta_{l|m}$ of the left hand side of (5.7). To compute the image of the right hand side, one also needs to use (5.35). Then one looks at the various matrix entries of the resulting equation to reduce to checking the following three identities



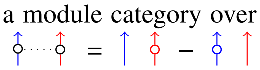
plus their images under the symmetry flip. To prove the first two of these, simplify them by multiplying the bottom left string with a clockwise internal bubble and using Lemma 5.5; the resulting identities then follow from Lemmas 5.9 and 5.10, respectively. For the final one, use

Lemma 5.11 to commute the clockwise internal bubble in the first diagram past the crossing below it, then use Lemma 5.5 and a commuting relation. □

6. A new proof of the basis theorem

By a *module category* over $\mathcal{H}eis_k$, we mean a $\Bbbk$-linear category $\mathcal{V}$ together with a $\Bbbk$-linear monoidal functor $\mathcal{H}eis_k \to \mathcal{E}nd_{\Bbbk}(\mathcal{V})$, where $\mathcal{E}nd_{\Bbbk}(\mathcal{V})$ denotes the strict $\Bbbk$-linear monoidal category consisting of $\Bbbk$-linear endofunctors and natural transformations. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are two $\Bbbk$-linear categories. Let $\mathcal{V} \boxtimes \mathcal{W}$ be the $\Bbbk$-linear category with objects that are pairs (X, Y) of objects $X \in \mathcal{V}$ and $Y \in \mathcal{W}$, and morphisms defined from

$$\mathrm{Hom}_{\mathcal{V}\boxtimes\mathcal{W}}((X_1, Y_1), (X_2, Y_2)) := \mathrm{Hom}_{\mathcal{V}}(X_1, X_2) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{W}}(Y_1, Y_2).$$

The rule for composition of morphisms in $\mathcal{V} \boxtimes \mathcal{W}$ is $(e \otimes f) \circ (g \otimes h) := (e \circ g) \otimes (f \circ h)$. If $\mathcal{V}$ and $\mathcal{W}$ are module categories over $\mathcal{H}eis_l$ and $\mathcal{H}eis_m$, respectively, then $\mathcal{V} \boxtimes \mathcal{W}$ is naturally a module category over the symmetric product $\mathcal{H}eis_l \odot \mathcal{H}eis_m$. If in addition the morphism  acts invertibly on all objects of $\mathcal{V} \boxtimes \mathcal{W}$, then this categorical action extends to an action of the localization $\mathcal{H}eis_l \,\bar{\odot}\, \mathcal{H}eis_m$ from Section 5. Hence, we can use the categorical comultiplication $\Delta_{l|m}$ from Theorem 5.3 to make $\mathcal{V} \boxtimes \mathcal{W}$ into a module category over $\mathcal{H}eis_k$ where $k = l + m$. In this section, we are going to use this idea to give an efficient proof of the basis theorem for the morphism spaces in $\mathcal{H}eis_k$ from [B, Theorem 1.6].

To get started, we need a source of Heisenberg module categories. These come from degenerate cyclotomic Hecke algebras. Assume that

$$f(w) = w^l + f_1 w^{l-1} + \cdots + f_l, \qquad\qquad g(w) = w^m + g_1 w^{m-1} + \cdots + g_m \tag{6.1}$$

are monic polynomials in $\Bbbk[w]$ of degrees $l, m \geq 0$. The *degenerate cyclotomic Hecke algebra* H_n^f associated to the polynomial $f(w)$ is the quotient $AH_n/(f(x_1))$; in case $n = 0$ we have that $H_0^f = AH_0 = \Bbbk$ by convention. This algebra has the well-known basis

$$\left\{ x_1^{a_1} \cdots x_n^{a_n} \pi \;\middle|\; \pi \in \mathfrak{S}_n, 0 \leq a_1, \dots, a_n < l \right\}; \tag{6.2}$$

see [Kl, Section 7.5]. In particular, one sees from this that the natural homomorphism $H_n^f \to H_{n+1}^f$ is injective. The following elementary lemma is well known; cf. [Kl, Proposition 2.2.2]. It implies that the eigenvalues of all x_i on any H_n^f-module lie in the same cosets of $\Bbbk$ modulo $\mathbb{Z}$ as the roots of the polynomial $f(w)$.

Lemma 6.1. *Assume that $\Bbbk$ is a field and V is a finite-dimensional AH_2-module. All eigenvalues of x_2 on V are of the form $\lambda, \lambda + 1$ or $\lambda - 1$ for eigenvalues λ of x_1 on V.*

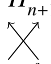
To the polynomials $f(w)$ and $g(w)$, we are going to associate a $\mathcal{H}eis_k$-module category $\mathcal{V}(f|g)$. As a $\Bbbk$-linear category, this is defined from $\mathcal{V}(f|g) := \mathcal{V}(f) \boxtimes \mathcal{V}(g)^\vee$ where

$$\mathcal{V}(f) := \bigoplus_{n\geq 0} H_n^f\text{-pmod}, \qquad \mathcal{V}(g)^\vee := \bigoplus_{n\geq 0} H_n^g\text{-pmod}. \tag{6.3}$$

To make $\mathcal{V}(f|g)$ into a module category, we first make $\mathcal{V}(f)$ and $\mathcal{V}(g)^\vee$ into $\mathcal{H}eis_{-l}$– and $\mathcal{H}eis_m$-module categories, respectively. According to [B, (1.23)], there is a strict $\Bbbk$-linear monoidal functor

$$\Psi_f : \mathcal{H}eis_{-l} \to \mathcal{E}nd_{\Bbbk}(\mathcal{V}(f)) \tag{6.4}$$

sending $\uparrow$ (resp., $\downarrow$) to the endofunctor defined on $M \in H_n^f$-pmod by the induction functor $\mathrm{ind}_n^{n+1} = H_{n+1}^f \otimes_{H_n^f} -$ (resp., the restriction functor res_{n-1}^n). On generating morphisms, Ψ_f sends

-  to the natural transformation defined on a projective H_n^f-module M by the map $H_{n+1}^f \otimes_{H_n^f} M \to H_{n+1}^f \otimes_{H_n^f} M, h \otimes v \mapsto hx_{n+1} \otimes v$;
-  to the natural transformation defined on a projective H_n^f-module M by the map $H_{n+2}^f \otimes_{H_n^f} M \to H_{n+2}^f \otimes_{H_n^f} M, h \otimes v \mapsto hs_{n+1} \otimes v$;

- and to the natural transformations defined by the unit and counit of the canonical adjunction making $(\mathrm{ind}_n^{n+1}, \mathrm{res}_n^{n+1})$ into an adjoint pair.

Thus we have made $\mathcal{V}(f)$ into a module category over $\mathcal{H}eis_{-l}$. Similarly, switching the roles of induction and restriction using $\jmath_n$ in place of $\imath_n$, we make $\mathcal{V}(g)^\vee$ into a $\mathcal{H}eis_m$-module category via the strict $\Bbbk$-linear monoidal functor

$$\Psi_g^\vee : \mathcal{H}eis_m \to \mathcal{E}nd_\Bbbk\left(\mathcal{V}(g)^\vee\right) \tag{6.5}$$

sending $\downarrow$ (resp., $\uparrow$) to the endofunctor defined on $M \in H_n^g$-pmod by the induction functor $\mathrm{ind}_n^{n+1} = H_{n+1}^g \otimes_{H_n^g} -$ (resp., the restriction functor res_{n-1}^n). On generating morphisms, $\Psi_g^\vee$ sends

- to the natural transformation defined on a projective H_n^g-module M by the map $H_{n+1}^g \otimes_{H_n^g} M \to H_{n+1}^f \otimes_{H_n^g} M, h \otimes v \mapsto hx_{n+1} \otimes v$;
- to the natural transformation defined on a projective H_n^g-module M by the map $H_{n+2}^g \otimes_{H_n^g} M \to H_{n+2}^g \otimes_{H_n^g} M, h \otimes v \mapsto -hs_{n+1} \otimes v$;
- and to the natural transformations defined by the unit and counit of the canonical adjunction making $(\mathrm{ind}_n^{n+1}, \mathrm{res}_n^{n+1})$ into an adjoint pair.

The proof of this is similar to the argument explained in [B, (1.23)], using one of the alternative presentations for $\mathcal{H}eis_m$ from Remark 5.2.

Lemma 6.2. *Suppose that $f(w) = (w - \lambda_1)\cdots(w - \lambda_l)$ and $g(w) = (w - \mu_1)\cdots(w - \mu_m)$ for $\lambda_i, \mu_j \in \Bbbk$ such that $\lambda_i - \mu_j \notin \mathbb{Z}$ for all i, j. In the categorical action of $\mathcal{H}eis_{-l} \odot \mathcal{H}eis_m$ on $\mathcal{V}(f|g) = \mathcal{V}(f) \boxtimes \mathcal{V}(g)^\vee$ arising from (6.6)–(6.7), acts invertibly on every object.*

Proof. Lemma 6.1 and the genericity assumption imply that the set of eigenvalues of $x_1, \dots, x_n$ on any finite-dimensional H_n^f-module is disjoint from the set of eigenvalues of $x_1, \dots, x_m$ on any finite-dimensional H_m^g-module. Consequently, the commuting endomorphisms defined by evaluating and on an object of $\mathcal{V}(f|g)$ have disjoint spectra. Hence, all eigenvalues of the endomorphism defined by = − lie in $\Bbbk^\times$. Consequently, this endomorphism is invertible. ∎

As explained in the opening paragraph of the section, it follows that there is a strict $\Bbbk$-linear monoidal functor $\mathcal{H}eis_{-l} \mathbin{\bar{\odot}} \mathcal{H}eis_m \to \mathcal{E}nd_\Bbbk(\mathcal{V}(f|g))$ for $f(w), g(w)$ satisfying the genericity assumption from Lemma 6.2. Composing with the categorical comultiplication $\Delta_{-l|m}$, we obtain a strict $\Bbbk$-linear monoidal functor

$$\Psi_{f|g} : \mathcal{H}eis_{m-l} \to \mathcal{E}nd_\Bbbk(\mathcal{V}(f|g))\,. \tag{6.6}$$

Thus we have made $\mathcal{V}(f|g)$ into a module category over $\mathcal{H}eis_{m-l}$.

Lemma 6.3. *In the categorical action of $\mathcal{H}eis_{m-l}$ on $\mathcal{V}(f|g)$ just defined, the generating functions (w) and (w) from (5.19)–(5.20) act on (H_0^f, H_0^g) by multiplication by $g(w)/f(w) \in w^{m-l}\Bbbk[\![w^{-1}]\!]$ and $f(w)/g(w) \in w^{l-m}\Bbbk[\![w^{-1}]\!]$, respectively.*

Proof. Applying [B, Lemma 1.8] with $g(w) = 1$, we get that

$$\Psi_f\left((w)\right)_{H_0^f} = f(w)^{-1}, \qquad \Psi_f\left((w)\right)_{H_0^f} = -f(w).$$

Similarly, applying it with $f(w) = 1$, we get that

$$\Psi_g^\vee\left((w)\right)_{H_0^g} = g(w), \qquad \Psi_g^\vee\left((w)\right)_{H_0^g} = -g(w)^{-1}.$$

Now use the identities (5.37)–(5.38). ∎

Now we can prove the basis theorem. To recall its statement, let $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ be objects of $\mathcal{H}eis_k$ for $X_i, Y_j \in \{\uparrow, \downarrow\}$. An *$(X,Y)$-matching* is a bijection between the sets $\{i \mid X_i = \uparrow\} \sqcup \{j \mid Y_j = \downarrow\}$ and $\{i \mid X_i = \downarrow\} \sqcup \{j \mid Y_j = \uparrow\}$. A *reduced lift* of an (X,Y)-matching means a diagram representing a morphism $X \to Y$ such that

- the endpoints of each string are points which correspond under the given matching;
- there are no floating bubbles and no dots on any string;
- there are no self-intersections of strings and no two strings cross each other more than once.

Fix a set $B(X,Y)$ consisting of a choice of reduced lift for each of the (X,Y)-matchings. Let $B_\circ(X,Y)$ be the set of all morphisms that can be obtained from the elements of $B(X,Y)$ by adding dots labelled with non-negative integer multiplicities near to the terminus of each string. Recall the homomorphism β from (5.22).

Theorem 6.4. *For $X, Y \in \mathcal{H}eis_k$, the morphism space* $\mathrm{Hom}_{\mathcal{H}eis_k}(X,Y)$ *is a free right* Sym-*module with basis $B_\circ(X,Y)$, where the right* Sym-*module structure is defined by $\phi\theta := \phi\otimes\beta(\theta)$.*

Proof. We just prove this when $k \leq 0$; the result for $k \geq 0$ then follows by applying Ω_k. Let $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ be two objects.

We first observe that $B_\circ(X,Y)$ spans $\mathrm{Hom}_{\mathcal{H}eis_k}(X,Y)$ as a right Sym-module. This is because there is a "straightening rule" allowing any diagram representing a morphism $X \to Y$ as a linear combination of the ones in $B_\circ(X,Y)$. This proceeds by induction on the number of crossings. Dots can be moved past crossings modulo a correction term with fewer crossings, so we can assume that all dots are at the termini of their strings. Also we can use the relations (5.3), (5.7), (5.23) and (5.24) to move strings into the same configuration as one of the chosen reduced lifts. Again this may produce correction terms with fewer crossings plus some floating bubbles. Finally floating bubbles can be moved to the right hand edge using (5.25), where they become scalars in Sym.

It remains to prove the linear independence. The main step is to do this in the special case that $X = Y = \uparrow^{\otimes n}$. Take a linear relation $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i) = 0$ for $\phi_i \in B_\circ(X,Y)$ and $\theta_i \in$ Sym. Choose $l \geq m \gg 0$ so that

- $k = m - l$;
- the multiplicities of dots in all ϕ_i arising in this linear relation are $< l$;
- all of the symmetric functions $\theta_i \in$ Sym are polynomials in the elementary symmetric functions $e_1, \dots, e_m$.

Let $u_1, \dots, u_m$ be indeterminates and let $\mathbb{K}$ be the algebraic closure of $\Bbbk(u_1, \dots, u_m)$. We are going to work now with algebras/categories that are linear over $\mathbb{K}$ (instead of the usual $\Bbbk$), adding a subscript $\mathbb{K}$ to our notation as we do to avoid any confusion. Consider the cyclotomic Hecke algebras ${}_{\mathbb{K}}H_n^f$ and ${}_{\mathbb{K}}H_n^g$ over $\mathbb{K}$ associated to the polynomials

$$f(w) := w^l, \qquad\qquad g(w) = w^m + u_1 w^{m-1} + \cdots + u_m.$$

Using the functor ${}_{\mathbb{K}}\Psi_{f|g}$ from (6.8), we make ${}_{\mathbb{K}}\mathcal{V}(f|g)$ into a ${}_{\mathbb{K}}\mathcal{H}eis_k$-module category. Since $\Bbbk \hookrightarrow \mathbb{K}$, there is a canonical $\Bbbk$-linear monoidal functor $\mathcal{H}eis_k \to {}_{\mathbb{K}}\mathcal{H}eis_k$, allowing us to view ${}_{\mathbb{K}}\mathcal{V}(f|g)$ also as a module category over $\mathcal{H}eis_k$. Now we evaluate the relation $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i) = 0$ on $({}_{\mathbb{K}}H_0^f, {}_{\mathbb{K}}H_0^g) \in {}_{\mathbb{K}}\mathcal{V}(f|g)$ to obtain a relation in ${}_{\mathbb{K}}H_n^f$. By the basis theorem for ${}_{\mathbb{K}}H_n^f$ from (6.3) and the choice of l, the images of $\phi_1, \dots, \phi_N$ in ${}_{\mathbb{K}}H_n^f$ are linearly independent over $\mathbb{K}$, so we deduce that the image of $\beta(\theta_i)$ in $\mathbb{K}$ is zero for each i. To deduce from this that $\theta_i = 0$, we know by the choice of m that θ_i is a polynomial in $e_1, \dots, e_m$. So we need to show that the images of $\beta(e_1), \dots, \beta(e_m)$ in $\mathbb{K}$ are algebraically independent. In fact, these images are the indeterminates $u_1, \dots, u_m$, respectively, as follows from Lemma 6.3 on noting that $g(w)/f(w) = w^k + u_1 w^{k-1} + \cdots + u_m w^{k-m}$.

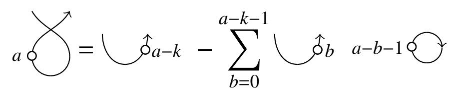
We have now proved the linear independence when $X = Y = \uparrow^{\otimes n}$. The general case reduces to this special case in just the same way as indicated in the proof of [Kh, Proposition 5]. Let us give some more details. First, we can use the canonical isomorphism $\mathrm{Hom}_{\mathcal{H}eis_k}(X,Y) \cong \mathrm{Hom}_{\mathcal{H}eis_k}(\mathbb{1}, X^* \otimes Y)$ arising from rigidity to reduce the proof of linear independence to the case that $X = \mathbb{1}$. Assume this from now on. The set $B_\circ(\mathbb{1}, Y)$ is empty unless Y has the same number n of $\uparrow$'s as $\downarrow$'s. Also we have already proved the linear independence in the case

$Y = \downarrow^{\otimes n} \otimes \uparrow^{\otimes n}$. So we may assume that Y has a subword $\uparrow \otimes \downarrow$. Let Z be Y with the two letters in the subword interchanged. By induction, we may assume the linear independence has already been established for $B_\circ(\mathbb{1}, Z)$. Now take a linear relation $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i)$ for $\phi_i \in B_\circ(\mathbb{1}, Y)$ and $\theta_i \in \mathrm{Sym}$. Recalling the isomorphism $\uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \xrightarrow{\sim} \downarrow \otimes \uparrow$ from (5.8), multiplying the subword $\uparrow \otimes \downarrow$ on top by the sideways crossing  defines a Sym-linear map

$$s : \mathrm{Hom}_{\mathcal{H}eis_k}(\mathbb{1}, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{H}eis_k}(\mathbb{1}, Z).$$

Unfortunately, s does not send $B_\circ(\mathbb{1}, Y)$ into $B_\circ(\mathbb{1}, Z)$, so we need to argue a little further. For $\phi \in B_\circ(\mathbb{1}, Y)$, there are three possibilities:

(1) If ϕ has a leftward cup labelled with a dots joining the letters in the subword then $s(\phi)$ has a dotted curl in this position, which can be rewritten using the relation



from (5.23). Thus $s(\phi) = \phi^\dagger + (*)$ where $\phi^\dagger$ is ϕ with the leftward cup labelled by a dots replaced with a rightward cup labelled by $a - k$ dots, and $(*)$ is a linear combination of similar-looking diagrams but with strictly fewer dots on the rightward cup and a clockwise bubble. This bubble may be moved to the right hand edge using (5.25), where it becomes a scalar in Sym; this process produces extra diagrams which have additional dots on the strings along the way. We may assume further that $B(\mathbb{1}, Z)$ was chosen so that $\phi^\dagger \in B_\circ(\mathbb{1}, Z)$. Let $B_1 = \bigcup_{b \geq 0} B_{1,b}$ where $B_{1,b}$ is the set of all $\psi \in B_\circ(\mathbb{1}, Z)$ which have a rightward cup labelled by a dot of multiplicity b joining the letters in the subword. Then we have shown that $s(\phi) = \phi^\dagger + (**)$ for $\phi^\dagger \in B_{1,a-k}$ and $(**)$ that is a linear combination of terms in $B_{1,b}$ for $0 \leq b < a - k$.

(2) If ϕ has two non-intersecting strings at the letters $\downarrow$ and $\uparrow$ of the subword, we can slide any dots on the $\uparrow$-string of $s(\phi)$ to the terminus to obtain $\phi^\dagger + (*)$ where $\phi^\dagger$ is a diagram that has intersecting strings at the letters of the subword, and $(*)$ is a linear combination of diagrams which have a dotted rightward cup at the subword. Again, we may assume that $\phi^\dagger \in B_\circ(\mathbb{1}, Z)$ by the choice of $B(\mathbb{1}, Z)$. Let B_2 be all elements of $B_\circ(\mathbb{1}, Z)$ with intersecting strings at the subword. Rewriting the error terms $(*)$ in terms of the basis, we deduce that $s(\phi) = \phi^\dagger + (**)$ for $\phi^\dagger \in B_2$ and $(**)$ that is a linear combination of terms in B_1.

(3) If ϕ has two intersecting strings at the letters $\downarrow$ and $\uparrow$, then $s(\phi)$ will have two strings that cross each other twice. Again, we slide dots to the terminus, producing also an error term $(*)$ which is a linear combination of terms in B_1. Then we use (5.7) (and possibly some other braid relations if there are other strings in between) to eliminate the crossings of the two strings in the leading term. Making a suitable choice of $B(\mathbb{1}, Z)$ and letting B_3 be the set of all elements of $B_\circ(\mathbb{1}, Z)$ with non-intersecting strings at the subword, we thus have that $s(\phi) = \phi^\dagger + (**)$ for $\phi^\dagger \in B_3$ and and $(**)$ that is a linear combination of terms in $B_1 \cup B_2$.

We have that $\sum s(\phi_i) \otimes \beta(\theta_i) = 0$. Ordering $B_\circ(\mathbb{1}, Z)$ so that $B_{1,0} < B_{1,1} < B_{1,2} < \cdots < B_2 < B_3$, we have shown that $s(\phi_i) = \phi_i^\dagger + (*)$ for $\phi_i^\dagger \in B_1 \cup B_2 \cup B_3$ and $(*)$ that is a linear combination of smaller $g \in B_1 \cup B_2 \cup B_3$. Also the elements $\phi_1^\dagger, \ldots, \phi_N^\dagger$ are all different. Hence, the known linear independence of $B_\circ(\mathbb{1}, Z)$ implies that $\theta_i = 0$ for all i, as required to complete the argument. □

Corollary 6.5. *The homomorphism* $\beta : \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k}(\mathbb{1})$ *is an isomorphism.*

Remark 6.6. As noted before Definition 5.1, the category $\mathcal{H}eis_k$ can be defined more generally over any commutative ground ring $\mathbb{k}$. Theorem 6.4 is easily extended to this situation: the proof of the spanning part of the result works for any $\mathbb{k}$; the linear independence in general may be deduced from the known linear independence over $\mathbb{Q}$ by standard base change arguments.

7. Proofs of Theorems 1.1 and 1.3

Recall the objects $S_\lambda^\pm \in \mathrm{Kar}(\mathcal{H}eis_k)$ for each $\lambda \in \mathcal{P}$ defined by (1.8). We note that

$$\Omega_k(S_\lambda^\pm) \cong S_{\lambda^T}^\mp, \tag{7.1}$$

with the transpose partition appearing because of the sign when Ω_k is applied to a crossing. The following provides the final important ingredient needed to prove the main results. The argument depends essentially on Theorems 2.2, 4.1 and 6.4.

Theorem 7.1. *The Grothendieck group $K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ is free as a $\mathbb{Z}$-module, with basis given by the elements $\{[S_\mu^- \otimes S_\lambda^+] \mid \lambda,\mu \in \mathcal{P}\}$ if $k \geq 0$ or $\{[S_\mu^+ \otimes S_\lambda^-] \mid \lambda,\mu \in \mathcal{P}\}$ if $k \leq 0$. Moreover, $\mathrm{Kar}(\mathcal{H}eis_k)$ is weakly cancellative in the sense that $[X] = 0 \Rightarrow X = 0$ for $X \in \mathrm{Kar}(\mathcal{H}eis_k)$.*

Proof. It suffices to treat the case $k \geq 0$; then the case $k \leq 0$ follows using (7.1). We make four elementary reductions which were suggested in [Kh, Section 5.1]:

(1) Let A be the locally unital algebra with distinguished idempotents $\{1_X \mid X \in \mathbb{A}\}$ that arises from the $\mathbb{k}$-linear category $\mathcal{H}eis_k$ as in (2.5). In view of the contravariant equivalence (2.6), it suffices to show that A is weakly cancellative and that $K_0(A)$ has basis $\{[Ae_{\mu|\lambda}] \mid \lambda,\mu \in \mathcal{P}\}$, where $e_{\mu|\lambda} := \jmath_{|\mu|}(e_\mu) \otimes \imath_{|\lambda|}(e_\lambda)$ for $\imath_n$ and $\jmath_n$ as in (5.16)–(5.17). Note these are the objects of A-pmod which correspond to the objects $S_{\mu^T}^- \otimes S_\lambda^+ \in \mathrm{Kar}(\mathcal{H}eis_k)$.

(2) For $d \in \mathbb{Z}$, let $\mathbb{A}_d$ be the set of all words $X \in \mathbb{A}$ such that the number of letters $\uparrow$ minus the number of letters $\downarrow$ is equal to d. Let

$$A^{(d)} := \bigoplus_{X,Y \in \mathbb{A}_d} 1_X A 1_Y.$$

Noting that $1_X A 1_Y = 0$ for $X \in \mathbb{A}_d$, $Y \in \mathbb{A}_e$ and $d \neq e$, we have that $A = \bigoplus_{d\in\mathbb{Z}} A^{(d)}$, hence, $K_0(A\text{-pmod}) = \bigoplus_{d\in\mathbb{Z}} K_0(A^{(d)}\text{-pmod})$. Therefore it is enough to show that $A^{(d)}$ is weakly cancellative and that $K_0(A^{(d)}\text{-pmod})$ is free with basis $\left\{[A^{(d)}e_{\mu|\lambda}] \;\middle|\; \lambda,\mu \in \mathcal{P}, |\lambda| - |\mu| = d\right\}$.

(3) Since $\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k}$, the left ideal $A^{(d)}1_X$ for $X \in \mathbb{A}_d$ is isomorphic to a direct sum of left ideals $A^{(d)}1_Y$ for words $Y \in \mathbb{A}_d$ in which all letters $\downarrow$ appear to the left of the letters $\uparrow$. Letting $\mathbb{A}_d^+$ denote the set of all such Y, this means that $A^{(d)}$ is Morita equivalent to the locally unital algebra

$$B^{(d)} := \bigoplus_{X,Y \in \mathbb{A}_d^+} 1_X A^{(d)} 1_Y.$$

Hence, we just need to show that $B^{(d)}$-pmod is weakly cancellative and that $K_0(B^{(d)}\text{-pmod})$ is free with basis $\left\{[B^{(d)}e_{\mu|\lambda}] \;\middle|\; \lambda,\mu \in \mathcal{P}, |\lambda| - |\mu| = d\right\}$.

(4) Next, we let $1_n^{(d)} := \sum_X 1_X$ summing over all words $X \in \mathbb{A}_d^+$ of length $\leq (2n + |d|)$. Then let

$$B_n^{(d)} := 1_n^{(d)} B^{(d)} 1_n^{(d)}.$$

This defines a direct system of locally unital algebras $0 = B_{-1}^{(d)} \subset B_0^{(d)} \subset B_1^{(d)} \subset \cdots$ whose union is $B^{(d)}$. Since any finitely generated B-module is a $B_n^{(d)}$-module for some n, the locally unital algebra B is weakly cancellative if $B_n^{(d)}$ is weakly cancellative for all n. Moreover,

$$K_0(B^{(d)}\text{-pmod}) = \varinjlim K_0(B_n^{(d)}\text{-pmod}).$$

We are now reduced to checking that $B_n^{(d)}$ is weakly cancellative and that $K_0(B_n^{(d)}\text{-pmod})$ is free on basis $\left\{[B_n^{(d)}e_{\mu|\lambda}] \;\middle|\; \lambda,\mu \in \mathcal{P}, |\lambda| - |\mu| = d, |\lambda| + |\mu| \leq 2n + |d|\right\}$.

To complete the proof of the theorem, we establish the truth of the statement just made by induction on $n = -1, 0, 1, \ldots$. For the induction step, take $n \geq 0$, set $R := B_n^{(d)}$ and $e := 1_{n-1}^{(d)}$. Note R is actually unital; its identity element is $1_n^{(d)}$. Also $eRe = B_{n-1}^{(d)}$, and by induction we may assume we have already proved that eRe-pmod is both weakly cancellative and free with basis $\left\{[B^{(d)}_{n-1}e_{\mu|\lambda}]\,\middle|\,\lambda,\mu\in\mathcal{P},|\lambda|-|\mu|=d,|\lambda|+|\mu|<2n+|d|\right\}$. Let $n_1,n_2\geq 0$ be defined from $n_1-n_2=d$ and $n_1+n_2=2n+|d|$. By Theorem 6.4, the quotient $S:=R/ReR$ has basis given by the elements $\pi(\phi\theta)$ for $\phi\in B_\circ(\downarrow^{\otimes n_2}\otimes\uparrow^{\otimes n_1},\downarrow^{\otimes n_2}\otimes\uparrow^{\otimes n_1})$ involving no cups or caps and θ running over a basis for Sym, where $\pi:R\twoheadrightarrow S$ is the quotient map. It follows that there is an isomorphism $AH_{n_1}\otimes_{\Bbbk}AH_{n_2}\otimes_{\Bbbk}\mathrm{Sym}\xrightarrow{\sim}S$, $\phi_1\otimes\phi_2\otimes\theta\mapsto\jmath_{n_2}(\phi_2)\otimes\imath_{n_1}(\phi_1)\otimes\beta(\theta)$. Moreover, $\sigma:S\to R,\pi(\phi\theta)\mapsto\phi\theta+e$ is a *unital* algebra homomorphism. Since we obviously have that $\pi\circ\sigma=\mathrm{id}_S$, this puts us in a position to apply Theorem 2.2. We deduce that the induction step follows from the assertions that $AH_{n_1}\otimes_{\Bbbk}AH_{n_2}\otimes_{\Bbbk}\mathrm{Sym}$ is weakly cancellative and $K_0(AH_{n_1}\otimes_{\Bbbk}AH_{n_2}\otimes_{\Bbbk}\mathrm{Sym})$ has basis $\{[AH_{n_1}e_\lambda\otimes_{\Bbbk}AH_{n_2}e_\mu\otimes_{\Bbbk}\mathrm{Sym}]\,|\,\lambda,\mu\in\mathcal{P},|\lambda|=n_1,|\mu|=n_2\}$. The first of these statements follows from Lemma 2.1, and the second from Theorem 4.1. $\square$

To prove Theorem 1.1, we are going to categorify some representations of Heis_k. The *basic representation* of Heis_{-1} is the ring $\mathrm{Sym}_{\mathbb{Z}}$ of symmetric funtions viewed as a Heis_{-1}-module so that for $f\in\mathrm{Sym}_{\mathbb{Z}}$ the element f^+ acts by left multiplication by f, and f^- acts by the adjoint operator with respect to the usual form $\langle -,-\rangle$ on $\mathrm{Sym}_{\mathbb{Z}}$, i.e., $\langle s_\lambda,s_\mu\rangle:=\delta_{\lambda,\mu}$. In particular, the generators of Heis_{-1} act on the basis of Schur functions as follows:

- $h_n^+s_\lambda=\sum s_\mu$ summing over all partitions μ whose Young diagram is obtained from that of λ by adding a box to the end of n different columns;
- $e_n^-s_\lambda=\sum s_\mu$ summing over partitions μ whose Young diagram is obtained from that of λ by removing a box from the end of n different rows.

Let $\mathrm{Sym}_{\mathbb{Z}}^\vee$ be the Heis_1-module obtained from $\mathrm{Sym}_{\mathbb{Z}}$ using $\omega_1:\mathrm{Heis}_1\xrightarrow{\sim}\mathrm{Heis}_{-1}$. Thus, denoting s_λ instead by $s_\lambda^\vee$ to avoid confusion, the action of Heis_1 on $\mathrm{Sym}_{\mathbb{Z}}^\vee$ satisfies

- $h_n^+s_\lambda^\vee=\sum s_\mu^\vee$ summing over all partitions μ whose Young diagram is obtained from that of λ by removing a box from the end of n different columns;
- $e_n^-s_\lambda^\vee=\sum s_\mu^\vee$ summing over partitions μ whose Young diagram is obtained from that of λ by adding a box to the end of n different rows.

More generally, for any $l,m\geq 0$ and $k:=m-l$, the tensor product $V(l|m):=\mathrm{Sym}_{\mathbb{Z}}^{\otimes l}\otimes\left(\mathrm{Sym}_{\mathbb{Z}}^\vee\right)^{\otimes m}$ is naturally a Heis_k-module. It has a natural monomial basis indexed by $(l+m)$-tuples of partitions. The associated representation

$$\psi_{l|m}:\mathrm{Heis}_k\to\mathrm{End}_{\mathbb{Z}}\left(V(l|m)\right) \tag{7.2}$$

is faithful as soon as $l+m>0$; the proof of faithfulness is particularly easy when both $l>0$ and $m>0$ which is all that we use below.

For monic $f(w)\in\Bbbk[w]$ of degree one, the inclusion $\Bbbk\mathfrak{S}_n\hookrightarrow H_n^f$ is actually an algebra isomorphism. Thus, the $\mathcal{H}eis_{-1}$-module category $\mathcal{V}(f)$ from (6.6) is the semisimple Abelian category $\bigoplus_{n\geq 0}\Bbbk\mathfrak{S}_n$-pmod, and there is an isomorphism $\mathrm{Sym}_{\mathbb{Z}}\xrightarrow{\sim}K_0(\mathcal{V}(f)),s_\lambda\mapsto[S(\lambda)]$ of $\mathbb{Z}$-modules. Similar statements hold for the $\mathcal{H}eis_1$-module category $\mathcal{V}(g)^\vee$ from (6.7) when $g(w)\in\Bbbk[w]$ is of degree one. More generally, for $u_1,\dots,u_l,v_1,\dots,v_m\in\Bbbk$, the category

$$\mathcal{V}(u_1,\dots,u_l|v_1,\dots,v_m):=\mathcal{V}(w-u_1)\boxtimes\cdots\boxtimes\mathcal{V}(w-u_l)\boxtimes\mathcal{V}(w-v_1)^\vee\boxtimes\cdots\boxtimes\mathcal{V}(w-v_m)^\vee$$

is a semisimple Abelian category, and there is a $\mathbb{Z}$-module isomorphism

$$V(l|m)\xrightarrow{\sim}K_0\left(\mathcal{V}(u_1,\dots,u_l|v_1,\dots,v_m)\right), \tag{7.3}$$

$$s_{\lambda^{(1)}}\otimes\cdots\otimes s_{\lambda^{(l)}}\otimes s^\vee_{\mu^{(1)}}\otimes\cdots\otimes s^\vee_{\mu^{(m)}}\mapsto\left[\left(S(\lambda^{(1)}),\dots,S(\lambda^{(l)}),S(\mu^{(1)}),\dots,S(\mu^{(m)})\right)\right].$$

This is a module category over $\mathcal{H}eis_{-1}\odot\cdots\odot\mathcal{H}eis_{-1}\odot\mathcal{H}eis_1\odot\cdots\odot\mathcal{H}eis_1$. If we assume in addition that $u_1,\dots,u_l,v_1,\dots,v_m$ are generic in the sense that their images in $\Bbbk/\mathbb{Z}$ are all different, then we can argue as in Lemma 6.2 to see that the action extends to the localization $\mathcal{H}eis_{-1}\,\overline{\odot}\,\cdots\,\overline{\odot}\,\mathcal{H}eis_{-1}\,\overline{\odot}\,\mathcal{H}eis_1\,\overline{\odot}\,\cdots\,\overline{\odot}\,\mathcal{H}eis_1$. Using the iterated categorical comultiplication from Theorem 5.3 (and the coassociativity noted in Remark 5.4), it becomes a module category over $\mathcal{H}eis_k$. Thus, there is a strict $\Bbbk$-linear monoidal functor

$$\Psi_{l|m} : \mathcal{H}eis_k \to \mathcal{E}nd_{\Bbbk}(\mathcal{V}(u_1,\dots,u_l|v_1,\dots,v_m)). \tag{7.4}$$

Since $\mathcal{V}(u_1,\dots,u_l|v_1,\dots,v_m)$ is Abelian, this extends to a functor from $\mathrm{Kar}(\mathcal{H}eis_k)$, which we denote by the same notation $\Psi_{l|m}$. The following shows that this functor categorifies (7.2).

Theorem 7.2. *There is a ring isomorphism $\gamma_k : \mathrm{Heis}_k \to K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ sending $s^\pm_\lambda \mapsto [S^\pm_\lambda]$ for each $\lambda \in \mathcal{P}$. Moreover, for generic $u_1,\dots,u_l,v_1,\dots,v_m$ with $k = m - l$, the diagram*

$$\begin{tikzcd} \mathrm{Heis}_k \arrow[r, "\psi_{l|m}"] \arrow[d, hook, two heads, "\gamma_k"'] & \mathrm{End}_{\mathbb{Z}}(V(l|m)) \arrow[d, hook, two heads, "c_k"] \\ K_0(\mathrm{Kar}(\mathcal{H}eis_k)) \arrow[r, "{[\Psi_{l|m}(-)]}"'] & \mathrm{End}_{\mathbb{Z}}(K_0(\mathcal{V}(u_1,\dots,u_l|v_1,\dots,v_m)) \end{tikzcd} \tag{7.5}$$

commutes, where c_k is the ring isomorphism defined by conjugating with (7.3), and the bottom map is the ring homomorphism $[X] \mapsto [\Psi_{l|m}(X)]$.

Proof. Theorem 7.1 shows there is a $\mathbb{Z}$-module isomorphism $\gamma_k : \mathrm{Heis}_k \xrightarrow{\sim} K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ sending $s^-_\mu s^+_\lambda \mapsto [S^-_\mu \otimes S^+_\lambda]$ if $k \geq 0$ or $s^+_\mu s^-_\lambda \mapsto [S^+_\mu \otimes S^-_\lambda]$ if $k \leq 0$, although we do not yet know that this is a ring homomorphism. Taking this as the definition of the left hand map, we are going to show in the next paragraph that the diagram (7.5) commutes for all generic u_i, v_j. This is all that is needed to complete the proof: since the top and bottom maps in (7.5) are ring homomorphisms, the right hand map is a ring isomorphism, and moreover $\psi_{l|m}$ is injective for any sufficiently large l and m, the commutativity of the diagram implies that the left hand map γ_k is a ring homomorphism too.

To see that the diagram commutes, it suffices to check that it commutes on each of the basis vectors via which γ_k has been defined. This reduces easily to checking that

$$c_k(s^\pm_\lambda v) = [S^\pm_\lambda]c_k(v) \tag{7.6}$$

for each $\lambda \in \mathcal{P}$ and $v \in V(l|m)$. The restrictions γ^-_k and γ^+_k of the map γ_k to the subalgebras $\mathrm{Heis}^-_k = \mathrm{Sym}_{\mathbb{Z}} \otimes 1$ and $\mathrm{Heis}^+_k = 1 \otimes \mathrm{Sym}_{\mathbb{Z}}$, respectively, are both ring homomorphisms. This follows for γ^+_k because $\gamma^+_k(s^+_\lambda) = [\imath](\gamma(s_\lambda))$, where γ is the ring isomorphism from (3.3) and $[\imath]$ is the ring homomorphism induced by the monoidal functor $\imath : \mathcal{S}ym \to \mathcal{H}eis_k$ defined just before (5.16). To see it for γ^-_k, use instead that $\gamma^-_k(s^-_\lambda) = [\jmath](\gamma(s_{\lambda^T}))$ for the monoidal functor $\jmath : \mathcal{S}ym \to \mathcal{H}eis_k$ arising from (5.17). In view of this and the fact that $\mathrm{Heis}^\pm_k$ is generated by $\{h^\pm_n \mid n \geq 1\}$, we deduce that (7.6) follows if we can establish just that

$$c_k(h^\pm_n v) = [H^\pm_n]c_k(v). \tag{7.7}$$

By the definition of (6.6), the object $H^+_n \in \mathrm{Kar}(\mathcal{H}eis_{-1})$ acts on $S(\lambda) \in \Bbbk\mathfrak{S}_m$-pmod by

$$H^+_n S(\lambda) = \mathrm{ind}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n} S(\lambda) \boxtimes \mathrm{triv}_n,$$

which is the image of $h_n s_\lambda$ under the isomorphism $\mathrm{Sym}_{\mathbb{Z}} \xrightarrow{\sim} K_0(\mathcal{V}(w - u_i))$. Thus h^+_n and $[H^+_n]$ act in the same way under this isomorphism. Since $H^-_n = (H^+_n)^*$, we deduce from this that h^-_n and $[H^-_n]$ act in the same way too. Similar statements hold for the action on $\mathrm{Sym}_{\mathbb{Z}} \cong [\mathcal{V}(w - v_j)^\vee]$ for each j. Recalling (1.2) and (1.7), $\psi_{l|m}(h^\pm_n)$ is multiplication by

$$\sum_{r_1+\cdots+r_{l+m}=n} h^\pm_{r_1} \otimes \cdots \otimes h^\pm_{r_{l+m}}.$$

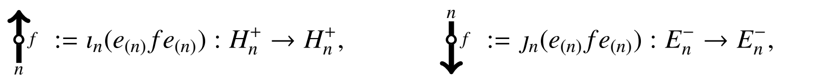
Now (7.7) follows from (5.36), as that shows that $\Psi_{l|m}(H^\pm_n)$ satisfies an analogous formula. □

Proof of Theorem 1.1. The isomorphism γ_k is constructed in Theorem 7.2. The weakly cancellative property is established in Theorem 7.1. □

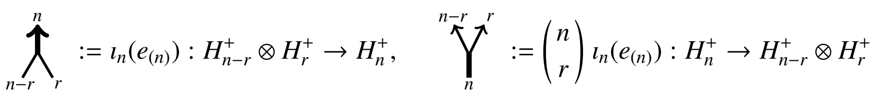
Proof of Theorem 1.3. As the maps involved are ring homomorphisms, it suffices to show that the diagram commutes on the generators h^+_n and e^-_n of Heis_k, which follows from (5.36). □

8. Proof of Theorem 1.2

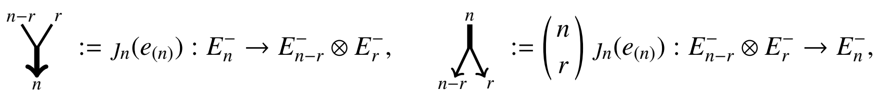
To prove Theorem 1.2, we need some explicit maps. To write these down, we use some "thick calculus" in the same spirit as [KLMS]. For $X, Y \in \mathcal{H}eis_k$ and idempotents $e_X : X \to X$ and $e_Y : Y \to Y$ we have that $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{H}eis_k)}((X, e_X), (Y, e_Y)) = e_Y \,\mathrm{Hom}_{\mathcal{H}eis_k}(X, Y) e_X$ by the definition of Karoubi envelope. We will denote the identity endomorphisms of the objects $H_n^+ = (\uparrow^{\otimes n}, \imath_n(e_{(n)}))$ and $E_n^- = (\downarrow^{\otimes n}, \jmath_n(e_{(n)}))$ by thick strings labelled by n, upward for H_n^+ and downward for E_n^-. We stress that these objects are *not* duals (unless $n = 1$). Instead, they are interchanged by the symmetry Ω_k. We introduce more diagrammatic shorthands:



$$:= \imath_n(e_{(n)} f e_{(n)}) : H_n^+ \to H_n^+, \qquad := \jmath_n(e_{(n)} f e_{(n)}) : E_n^- \to E_n^-, \tag{8.1}$$

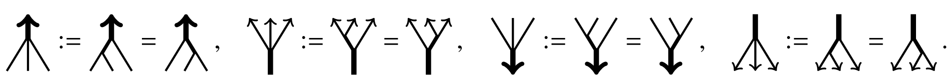


$$:= \imath_n(e_{(n)}) : H_{n-r}^+ \otimes H_r^+ \to H_n^+, \qquad := \binom{n}{r} \imath_n(e_{(n)}) : H_n^+ \to H_{n-r}^+ \otimes H_r^+, \tag{8.2}$$



$$:= \jmath_n(e_{(n)}) : E_n^- \to E_{n-r}^- \otimes E_r^-, \qquad := \binom{n}{r} \jmath_n(e_{(n)}) : E_{n-r}^- \otimes E_r^- \to E_n^-, \tag{8.3}$$

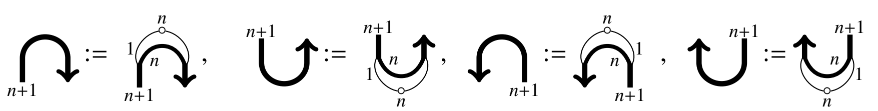
for $0 \leq r \leq n$ and $f \in \mathrm{Sym}_n$. Again, the downward morphisms in (8.1)–(8.3) are the images of the upward ones under Ω_k. Also, the merge and split morphisms are associative in an obvious sense allowing their definition to be extended to more strings, e.g., for three strings:



The identities (4.22)–(4.23) imply for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ that

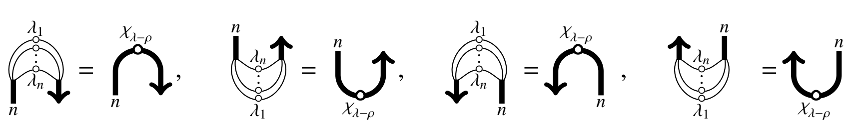
$$= \sum_{\pi \in \mathfrak{S}_n} \pi \oplus x^\lambda, \qquad = \sum_{\pi \in \mathfrak{S}_n} \pi \ominus x^\lambda, \tag{8.4}$$

where $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. There are thick cups and caps, which we define recursively by setting



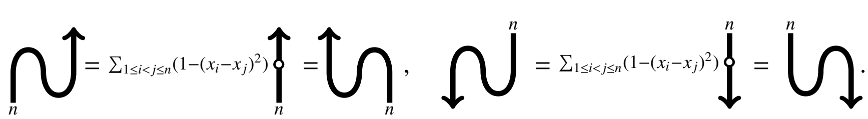
$$\tag{8.5}$$

Symmetric polynomials commute across thick cups and caps, so we may also draw them at the critical point without ambiguity. By (4.25), we have for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ that



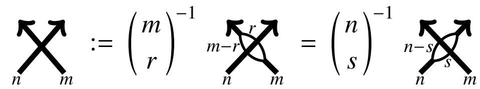
$$\tag{8.6}$$

where $\mathcal{X}_\lambda$ is the (signed) Schur polynomial from (4.24) and $\rho = (n-1, \cdots, 1, 0) \in \mathbb{Z}^n$. We note also that

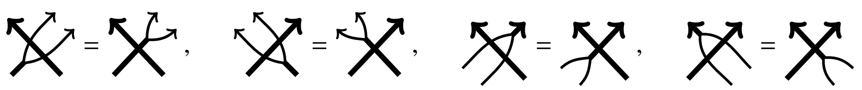


$$= \sum_{1 \leq i < j \leq n} (1 - (x_i - x_j)^2), \qquad = \sum_{1 \leq i < j \leq n} (1 - (x_i - x_j)^2) \tag{8.7}$$

Finally, there are thick upward crossings which are defined recursively so that

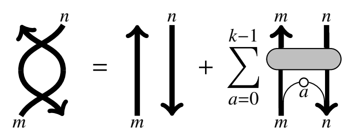

(8.8)

for any $0 < r < m, 0 < s < n$. There are thickened versions of the braid and quadratic relations (3.1). Moreover, the braid relation implies further relations:


(8.9)

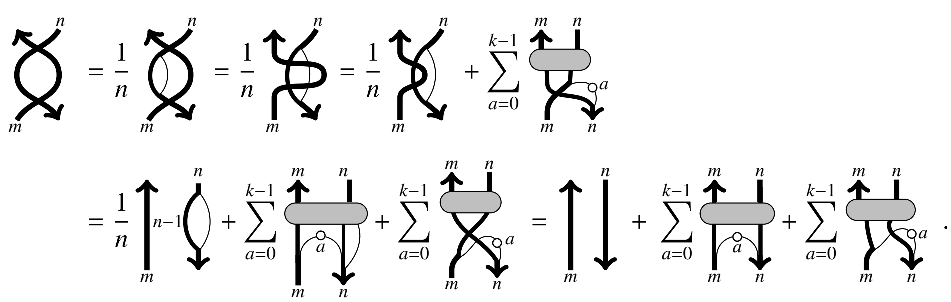
Similarly, there are thick downward, rightward and leftward crossings, defined by the same pictures as (8.8) with different orientations. Analogs of (8.9) hold for all possible orientations.

Lemma 8.1. *Assume that $k \geq 0$ and $m, n > 0$. Then*



where a shaded box indicates a morphism which will not be determined precisely.

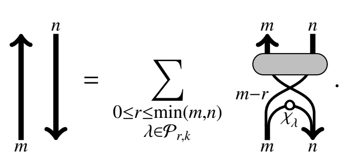
Proof. We proceed by induction on n. The base case will be discussed in the next paragraph. For the induction step, assuming $n > 1$, the induction hypothesis gives us that



Now we commute the a dots in the final term to the left past the crossing. This also produces correction terms, but these all have strictly fewer than a dots on the cap so are allowed.

It just remains to treat the base case $n = 1$. This proceeds by induction on $m = 1, 2, \ldots$. The case $m = 1$ follows from (5.7) and (5.5). The induction step follows by a calculation which is the mirror image in a vertical axis of the calculation in the previous paragraph, starting by splitting the string of thickness m into strings of thickness 1 and $m - 1$. □

Corollary 8.2. *For $k \geq 0$ and $m, n > 0$, we have that*

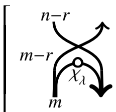


Proof. Rearrange the identity from Lemma 8.1 to get the $r = 0$ term in the sum directly, then use induction on $\min(m, n)$ plus (8.6) to get the other terms. □

Proof of Theorem 1.2. We just treat the case $k \geq 0$; the result for $k \leq 0$ then follows easily by applying Ω_k (also transposing matrices). The thick upward (resp., downward) crossing gives a canonical isomorphism $H_m^+ \otimes H_n^+ \xrightarrow{\sim} H_n^+ \otimes H_m^+$ (resp., $E_m^- \otimes E_n^- \xrightarrow{\sim} E_n^- \otimes E_m^-$). For the remaining relation, we must construct an isomorphism between the objects

$$P := H_m^+ \otimes E_n^-, \qquad\qquad Q := \bigoplus_{r=0}^{\min(m,n)} \bigoplus_{\lambda \in \mathcal{P}_{r,k}} E_{n-r}^- \otimes H_{m-r}^+.$$

Corollary 8.2 shows that the morphism $\theta_{m,n} : P \to Q$ defined by the column vector



$$\left[\quad\right]_{0\leq r\leq \min(m,n), \lambda\in\mathcal{P}_{r,k}} \tag{8.10}$$

has a left inverse $\phi_{m,n}$. Moreover, thanks to Theorem 1.1 and (1.6), we have that $[P] = [Q]$ in $K_0(\mathrm{Kar}(\mathcal{H}eis_k))$. In view of the weakly cancellative property, this is enough to imply that $\phi_{m,n}$ is actually the two-sided inverse of $\theta_{m,n}$.

To explain the last assertion in more detail, we use (2.6) to translate into a statement about projective modules over the locally unital algebra A arising from $\mathcal{H}eis_k$. Remembering that this is a contravariant equivalence, we have finitely generated projective A-modules P, Q such that $[P] = [Q]$, and homomorphisms $\theta_{m,n} : Q \to P$ and $\phi_{m,n} : P \to Q$ such that $\theta_{m,n} \circ \phi_{m,n} = \mathrm{id}_P$, and need to show that $\theta_{m,n}$ is an isomorphism. Let $R := \ker \theta_{m,n}$. Since $\theta_{m,n}$ has a right inverse, it is surjective. Since Q is projective, we have that $Q \cong P \oplus R$. Since $[P] = [Q]$, we deduce that $[R] = 0$. Hence, $R = 0$ as A is weakly cancellative. □

References

[B] J. Brundan, On the definition of Heisenberg category, *Alg. Comb.* **1** (2018), 523–544.

[BCNR] J. Brundan, J. Comes, D. Nash and A. Reynolds, A basis theorem for the affine oriented Brauer category and its cyclotomic quotients, *Quantum Topology* **8** (2017), 75–112.

[BSW] J. Brundan, A. Savage and B. Webster, On the definition of quantum Heisenberg category, preprint.

[Kh] M. Khovanov, Heisenberg algebra and a graphical calculus, *Fund. Math.* **225** (2014), 169–210.

[KLMS] M. Khovanov, A. Lauda, M. Mackaay and M. Stŏsić, Extended graphical calculus for categorified quantum $\mathfrak{sl}(2)$, *Mem. Amer. Math. Soc.* **219** (2012), no. 1029, 87pp..

[Kl] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge University Press, 2005.

[LRS] A. Licata, D. Rosso, and A. Savage, A graphical calculus for the Jack inner product on symmetric functions, *J. Combin. Theory Ser. A* **155** (2018), 503–543.

[M] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, second edition, OUP, 1995.

[MS] M. Mackaay and A. Savage, Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification, *J. Algebra* **505** (2018), 150–193.

[Q] D. Quillen, Higher algebraic K-theory I, *Lect. Notes Math.* **341** (1973), 85-147.

[R] J. Rosenberg, *Algebraic K-Theory and its Applications*, Graduate Texts in Mathematics, vol. 147, Springer-Verlag, 1994.

[S] D. B. Suarez, Integral presentations of quantum lattice Heisenberg algebras, in: "Categorification and Higher Representation Theory," *Contemp. Math.*, vol. 683, pp. 247–259, Amer. Math. Soc., Providence, RI, 2017.

[TV] V. Turaev and A. Virelizier, *Monoidal Categories and Topological Field Theory*, Progress in Mathematics, 322, Birkhuser/Springer, 2017.

[W] B. Webster, Unfurling Khovanov-Lauda-Rouquier algebras; `arXiv:1603.06311`.

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