ON THE DEFINITION OF KAC-MOODY 2-CATEGORY

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Abstract. We show that the Kac-Moody 2-categories defined by Rouquier and by Khovanov and Lauda are the same.

1. Introduction

Assume that we are given the following data:

- a (not necessarily finite) index set I;
- integers a_{ij} for each $i, j \in I$ such that $a_{ii} = 2$, $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Thus $A = (a_{ij})_{i,j \in I}$ is a generalized Cartan matrix. Set $d_{ij} := -a_{ij}$ for short. Fix also the additional data:

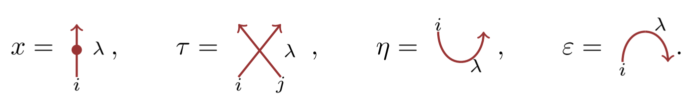
- a complex vector space $\mathfrak{h}$;
- linearly independent vectors $\alpha_i \in \mathfrak{h}^*$ for each $i \in I$ called *simple roots*;
- linearly independent vectors $h_i \in \mathfrak{h}$ for each $i \in I$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$.

Let $\mathfrak{g}$ be the associated Kac-Moody algebra with Chevalley generators $\{e_i, h_i, f_i\}_{i \in I}$ and Cartan subalgebra $\mathfrak{h}$. Let $P := \{\lambda \in \mathfrak{h}^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for all } i \in I\}$ be its *weight lattice*. In [R], Rouquier has associated to $\mathfrak{g}$ a certain 2-category $\mathfrak{A}(\mathfrak{g})$, which we will denote instead by $\mathcal{U}(\mathfrak{g})$. It depends also on

- a commutative ground ring $\Bbbk$;
- units $t_{ij} \in \Bbbk^\times$ for $i, j \in I$ such that $t_{ii} = 1$ and $t_{ij} = t_{ji}$ if $d_{ij} = 0$;
- scalars $s_{ij}^{pq} \in \Bbbk$ for $i, j \in I$ and $0 \leq p < d_{ij}, 0 \leq q < d_{ji}$ such that $s_{ij}^{pq} = s_{ji}^{qp}$.

The following is the definition from [R, §4.1.3] formulated in diagrammatic terms.

Definition 1.1. The *Kac-Moody 2-category* $\mathcal{U}(\mathfrak{g})$ is the strict additive $\Bbbk$-linear 2-category with object set P, generating 1-morphisms $E_i 1_\lambda : \lambda \to \lambda + \alpha_i$ and $F_i 1_\lambda : \lambda \to \lambda - \alpha_i$ for each $i \in I$ and $\lambda \in P$, and generating 2-morphisms $x : E_i 1_\lambda \to E_i 1_\lambda, \tau : E_i E_j 1_\lambda \to E_j E_i 1_\lambda$, $\eta : 1_\lambda \to F_i E_i 1_\lambda$ and $\varepsilon : E_i F_i 1_\lambda \to 1_\lambda$, subject to certain relations. To record these, we adopt a diagrammatic formalism like in [KL], representing the identity 2-morphisms of $E_i 1_\lambda$ and $F_i 1_\lambda$ by ${}^{\lambda+\alpha_i}\!\underset{i}{\uparrow}{}^{\lambda}$ and ${}^{\lambda-\alpha_i}\!\underset{i}{\downarrow}{}^{\lambda}$, respectively, and the other generators by



(1.1)

We stress that our diagrams are simply shorthands for algebraic expressions built by horizontally and vertically composing generators; they do not satisfy any topological invariance other than the "rectilinear isotopy" implied by the interchange

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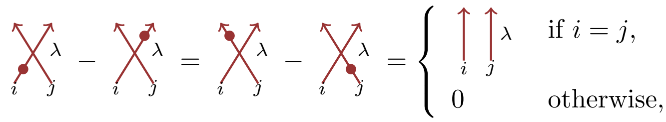
law. First, we have the *quiver Hecke relations*:



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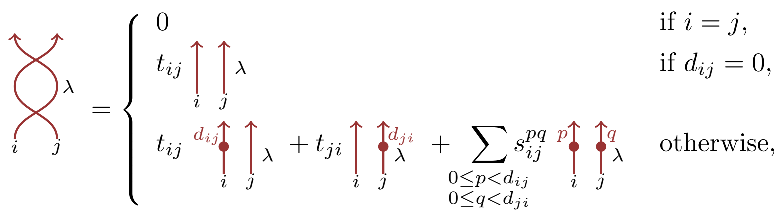
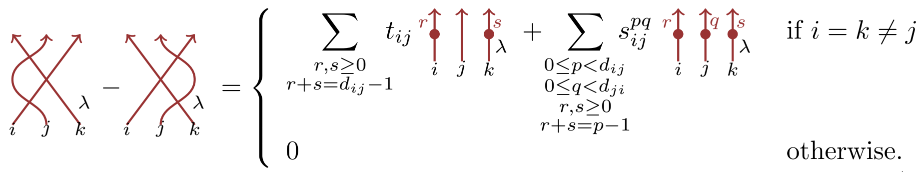
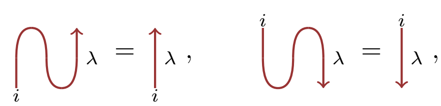


(1.3)



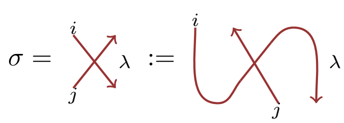
(1.4)

(Note in the above relations that we represent powers of x by decorating the dot with a multiplicity.) Next we have the *right adjunction relations*



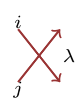
(1.5)

which imply that $F_i 1_{\lambda+\alpha_i}$ is the right dual of $E_i 1_\lambda$. Finally there are some *inversion relations*. To formulate these, we first introduce a new 2-morphism

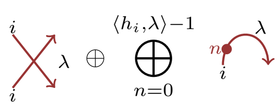


$: E_j F_i 1_\lambda \to F_i E_j 1_\lambda.$ (1.6)

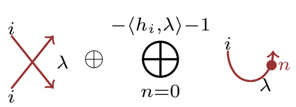
Then we require that the following 2-morphisms are isomorphisms:



$: E_j F_i 1_\lambda \xrightarrow{\sim} F_i E_j 1_\lambda$ if $i \neq j$, (1.7)



$: E_i F_i 1_\lambda \xrightarrow{\sim} F_i E_i 1_\lambda \oplus 1_\lambda^{\oplus\langle h_i,\lambda\rangle}$ if $\langle h_i, \lambda\rangle \geq 0$, (1.8)



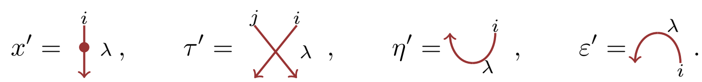
$: E_i F_i 1_\lambda \oplus 1_\lambda^{\oplus-\langle h_i,\lambda\rangle} \xrightarrow{\sim} F_i E_i 1_\lambda$ if $\langle h_i, \lambda\rangle \leq 0$. (1.9)

(This means formally that there are some additional as yet unnamed generators which serve as two-sided inverses to the 2-morphisms in (1.7)–(1.9).)

Our main theorem identifies the 2-category $\mathcal{U}(\mathfrak{g})$ just defined with the Khovanov-Lauda 2-category from [KL]. Actually Khovanov and Lauda worked just with the choice of parameters in which $t_{ij} = 1$ and $s_{ij}^{pq} = 0$ always. Subsequently, Cautis and Lauda [CL] generalized the definition to incorporate more general choices of these parameters as above. By the "Khovanov-Lauda 2-category" we really mean the more general version from [CL].

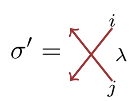
Main Theorem. *Rouquier's Kac-Moody* 2*-category* $\mathcal{U}(\mathfrak{g})$ *is isomorphic to the Khovanov-Lauda* 2*-category.*

The proof is an elementary relation chase. To explain the strategy, recall that the Khovanov-Lauda 2-category has the same objects and 1-morphisms as $\mathcal{U}(\mathfrak{g})$. Then there are generating 2-morphisms represented by the same diagrams as x, τ, η and ε above, plus additional generating 2-morphisms $x' : F_i 1_\lambda \to F_i 1_\lambda$, $\tau' : F_i F_j 1_\lambda \to F_j F_i 1_\lambda$, $\eta' : 1_\lambda \to E_i F_i 1_\lambda$ and $\varepsilon' : F_i E_i 1_\lambda \to 1_\lambda$ represented diagrammatically by


(1.10)

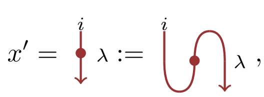
These satisfy further relations which we will recall in more detail later in the introduction. It is evident that all of the defining relations of $\mathcal{U}(\mathfrak{g})$ recorded above are satisfied in the Khovanov-Lauda 2-category. Hence there is a strict $\Bbbk$-linear 2-functor from $\mathcal{U}(\mathfrak{g})$ to the Khovanov-Lauda 2-category which is the identity on objects and 1-morphisms, and maps the generating 2-morphisms x, τ, η and ε to the corresponding 2-morphisms from [KL, CL].

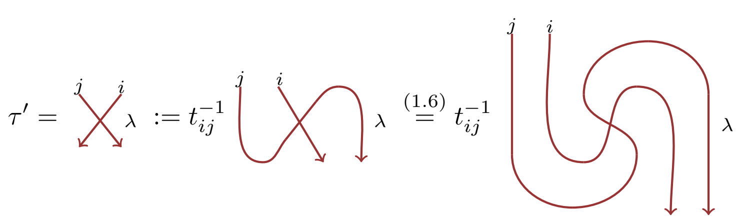
To see that this functor is an isomorphism, we construct a two-sided inverse. In order to do this, we need to identify appropriate 2-morphisms x', τ', η' and ε' in $\mathcal{U}(\mathfrak{g})$ that will be the images of the additional generators (1.10) under the inverse functor. The definitions of η' and ε' that follow are essentially the same as Rouquier's "candidates" for second adjunction from [R, §4.1.4], except that we have renormalized by the sign $(-1)^{\langle h_i, \lambda\rangle + 1}$ in order to be consistent with the conventions of [KL, CL]. We will also define a *leftward crossing*


$: F_i E_j 1_\lambda \to E_j F_i 1_\lambda,$ (1.11)

which we have chosen to normalize differently from the leftward crossing in [CL].

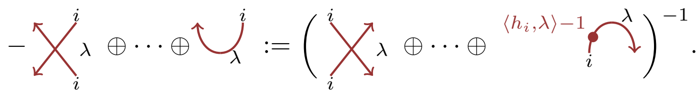
Definition 1.2. Define the downward dots and crossings x' and τ' to be the right mates of x and τ (up to the factor t_{ij}^{-1} in the latter case):


(1.12)

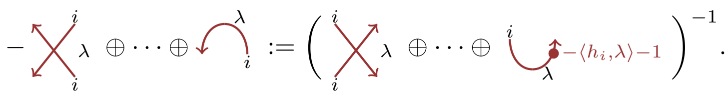

(1.13)

Then to define η', ε' and σ', we assume initially that $\langle h_i, \lambda\rangle > 0$. Thinking of (1.8) as a column vector of morphisms, its inverse is a row vector. We define the

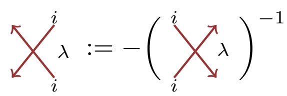
2-morphisms σ' and η' so that $-\sigma'$ is the leftmost entry of this row vector and η' is its rightmost entry:

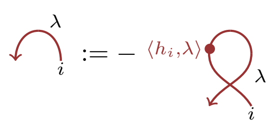
 (1.14)

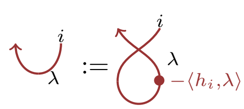
Instead, if $\langle h_i, \lambda\rangle < 0$, the morphism (1.9) is a row vector and its inverse is a column vector. We define σ' and ε' so that $-\sigma'$ is the top entry of this column vector and ε' is its bottom one:

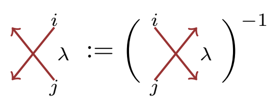
 (1.15)

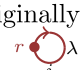
To complete the definitions of σ', η' and ε', it remains to set

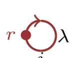
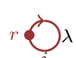
 if $\langle h_i, \lambda\rangle = 0$, (1.16)

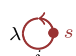
 if $\langle h_i, \lambda\rangle \geq 0$, (1.17)

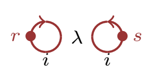
 if $\langle h_i, \lambda\rangle \leq 0$, (1.18)

 if $i \neq j$. (1.19)

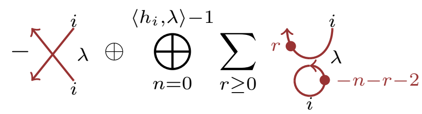
Now to prove the Main Theorem we must show that all of the defining relations for the Khovanov-Lauda 2-category from [CL] are satisfied by the 2-morphisms in $\mathcal{U}(\mathfrak{g})$ just introduced. First we will show that the *infinite Grassmannian relation* holds in $\mathcal{U}(\mathfrak{g})$. This relation was formulated originally by Lauda in [L]. It asserts that, as well as the dotted bubble 2-morphisms  ,  $\in \mathrm{End}(1_\lambda)$ already defined for $r, s \geq 0$, there are unique dotted bubble 2-morphisms for $r, s < 0$ such that

 $= 0$ if $r < \langle h_i, \lambda\rangle - 1$,  $= 1_{1_\lambda}$ if $r = \langle h_i, \lambda\rangle - 1$, (1.20)

 $= 0$ if $s < -\langle h_i, \lambda\rangle - 1$,  $= 1_{1_\lambda}$ if $s = -\langle h_i, \lambda\rangle - 1$, (1.21)

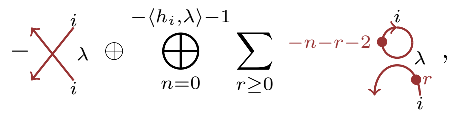
$$\sum_{\substack{r,s\in\mathbb{Z}\\ r+s=t-2}}$$  $= 0$ for all $t > 0$. (1.22)

Using the infinite Grassmannian relation, we deduce that the inverses of the 2-morphisms from (1.8) and (1.9) are



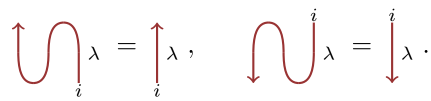
(1.23)

and



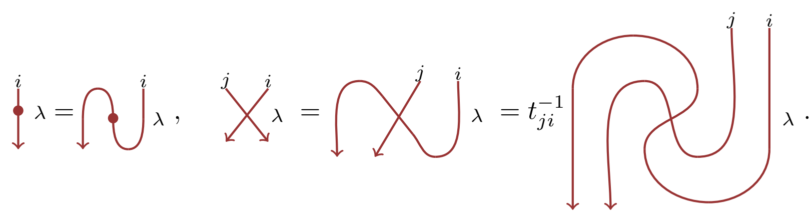
(1.24)

respectively. Several other of the Khovanov-Lauda relations follow from this assertion; see (3.16)–(3.18) below. After that, we will show that the 2-morphisms η' and ε' define a unit and a counit making the right dual $F_i 1_{\lambda+\alpha_i}$ of $E_i 1_\lambda$ also into its left dual:



(1.25)

Consequently, $\mathcal{U}(\mathfrak{g})$ is *rigid*, i.e. all of its 1-morphisms admit both a left and a right dual. Finally we will show that the other generating 2-morphisms are cyclic (up to scalars):



(1.26)

Acknowledgements. The diagrams in this paper were created using Till Tantau's `TikZ` package.

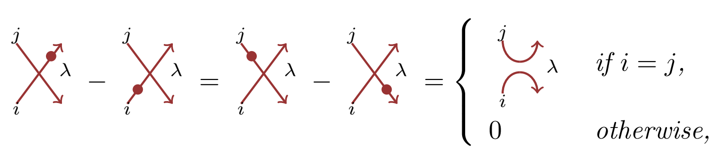
2. Chevalley involution

In this section we introduce a useful symmetry. First though we record some of the most basic additional relations that hold in the 2-category $\mathcal{U}(\mathfrak{g})$.

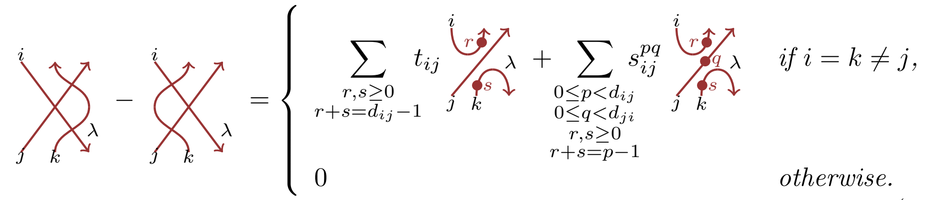
Lemma 2.1. *The following relations hold:*

(2.1)

(2.2)



(2.3)



(2.4)

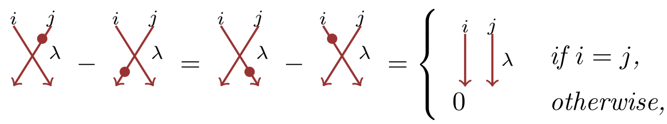
Proof. The first two relations follow from the definition (1.12) of the downward dot using the adjunction relations (1.5). The second two follow similarly from the definition of the rightward crossing (1.6). For (2.3), attach a rightward cap to the top right strand and a rightward cup to the bottom left strand of (1.2), then use (2.1) and the definition (1.6). Finally for (2.4), attach a rightward cap to the top right strand and a rightward cup to the bottom left strand in (1.4), then use (2.2) and the definition of the rightward crossing. □

Taking notation from [CL], we define new parameters from

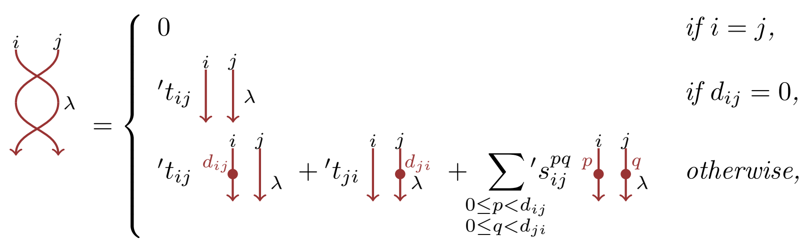
$$'t_{ij} := t_{ji}^{-1}, \qquad 's_{ij}^{pq} := t_{ij}^{-1} t_{ji}^{-1} s_{ji}^{qp}. \tag{2.5}$$

The next lemma explains the significance of these scalars.

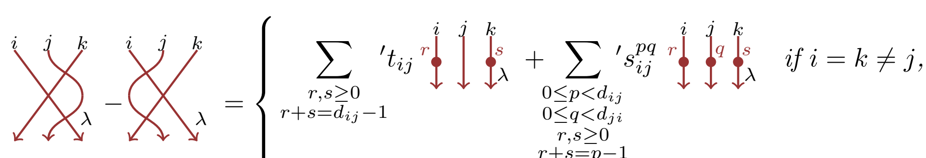
Lemma 2.2. *The following relations hold:*



(2.6)



(2.7)



(2.8)

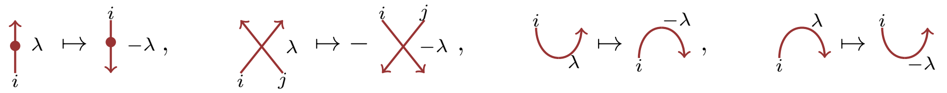
Proof. Put rightward caps on the top and rightward cups on the bottom of the relations (1.2)–(1.4), then use (1.5), the definitions (1.6), (1.12), (1.13), and (2.1)–(2.2). □

For any strict $\Bbbk$-linear 2-category $\mathcal{C}$, we write $\mathcal{C}^{\mathrm{opp}}$ for the 2-category with the same objects as $\mathcal{C}$ but with morphism categories defined from $\mathcal{H}om_{\mathcal{C}^{\mathrm{opp}}}(\lambda,\mu) := \mathcal{H}om_{\mathcal{C}}(\lambda,\mu)^{\mathrm{opp}}$, so the vertical composition in $\mathcal{C}^{\mathrm{opp}}$ is the opposite of the one in $\mathcal{C}$, while the horizontal composition in $\mathcal{C}^{\mathrm{opp}}$ is the same as in $\mathcal{C}$.

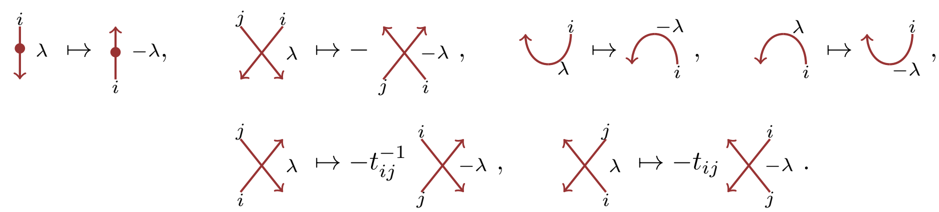
Theorem 2.3. *Let $'\mathcal{U}(\mathfrak{g})$ be the Kac-Moody 2-category defined as in Definition 1.1 but using the primed parameters from (2.5) in place of t_{ij} and s_{ij}^{pq}. Then there is an isomorphism of strict $\Bbbk$-linear 2-categories*

$$\mathtt{T} : {'\mathcal{U}(\mathfrak{g})} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})^{\mathrm{opp}}$$

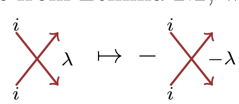
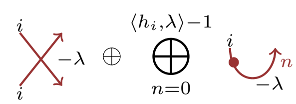
defined on objects by $\mathtt{T}(\lambda) := -\lambda$, on generating 1-morphisms by $\mathtt{T}(E_i 1_\lambda) := F_i 1_{-\lambda}$ and $\mathtt{T}(F_i 1_\lambda) := E_i 1_{-\lambda}$, and on generating 2-morphisms by



The effect of $\mathtt{T}$ on the other named 2-morphisms in $'\mathcal{U}(\mathfrak{g})$ is as follows:



Moreover we have that $\mathtt{T} \circ \mathtt{T}' = \mathrm{id} = \mathtt{T}' \circ \mathtt{T}$ where $\mathtt{T}' : \mathcal{U}(\mathfrak{g}) \xrightarrow{\sim} {'\mathcal{U}(\mathfrak{g})}^{\mathrm{opp}}$ is the analog of $\mathtt{T}$ with $\mathcal{U}(\mathfrak{g})$ replaced by $'\mathcal{U}(\mathfrak{g})$ and $'\mathcal{U}(\mathfrak{g})$ replaced by $''\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})$.

Proof. To see that $\mathtt{T}$ is well defined we need to verify that the images under $\mathtt{T}$ of the relations (1.2)–(1.5) and (1.7)–(1.9) with primed parameters hold in $\mathcal{U}(\mathfrak{g})^{\mathrm{opp}}$. For the first three, this follows from Lemma 2.2, while (1.5) is clear. For the remaining ones, we first note that  by the definition (1.6). Then for example for (1.8), we must show for $\langle h_i, \lambda\rangle \geq 0$ that  is invertible in $\mathcal{U}(\mathfrak{g})$. This follows by composing (1.9) with $\mathrm{diag}(-1, 1, \dots, 1)$, using also (2.1). The rest of the theorem is a routine check from the definitions (1.12)–(1.19). $\square$

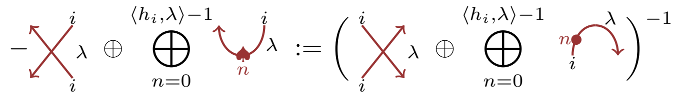
We will often appeal to Theorem 2.3 to establish mirror images of relations in a horizontal axis. For example, applying it to the analog of the relation (2.2) in $'\mathcal{U}(\mathfrak{g})$, we obtain the following relation (which could also be deduced directly from the definition of the downward crossing):

Corollary 2.4. *The following relations hold in $\mathcal{U}(\mathfrak{g})$:*

 (2.9)

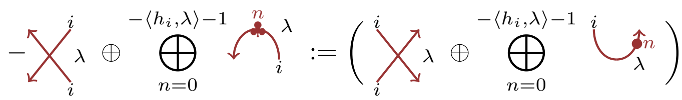
3. The infinite Grassmannian relation

The goal in this section is to show that the infinite Grassmannian relation holds in $\mathcal{U}(\mathfrak{g})$. We begin by introducing notation for the other entries of the row vector that is the two-sided inverse of (1.8) and of the column vector that is the two-sided inverse of (1.9): let



if $\langle h_i, \lambda\rangle \geq 0$,

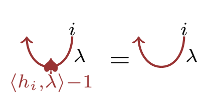
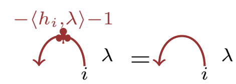
(3.1)



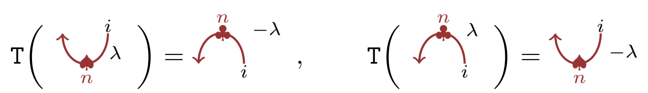
if $\langle h_i, \lambda\rangle \leq 0$.

(3.2)

We will give a more explicit description of these 2-morphisms in (3.12)–(3.13) below. Note right away comparing the present definitions with (1.14)–(1.15) that

 if $\langle h_i, \lambda\rangle > 0$,  if $\langle h_i, \lambda\rangle < 0$. (3.3)

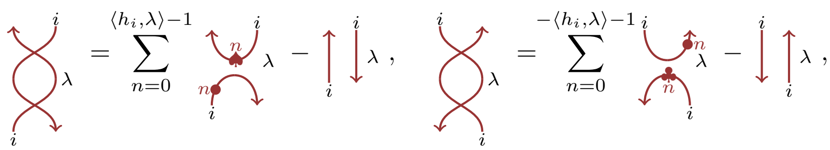
Also for all admissible values of n we have that



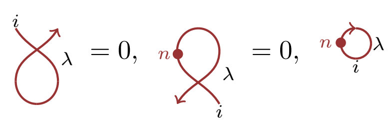
(3.4)

as follows on applying the isomorphism $\mathtt{T}$ from Theorem 2.3 to the definitions (3.1)–(3.2) and using (2.1).

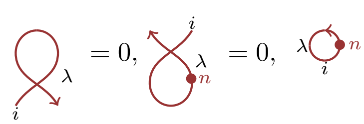
Lemma 3.1. *The following relations hold:*



(3.5)

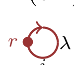
 $= \delta_{n,\langle h_i,\lambda\rangle-1} 1_{1_\lambda}$ *all assuming* $0 \leq n < \langle h_i, \lambda\rangle$,

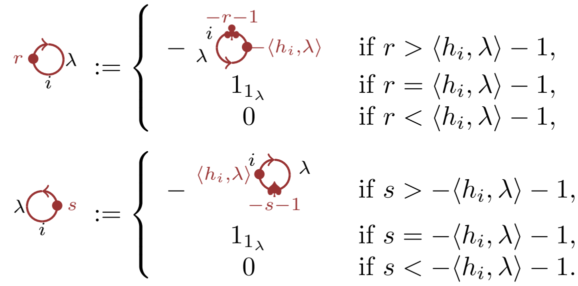
(3.6)

 $= \delta_{n,-\langle h_i,\lambda\rangle-1} 1_{1_\lambda}$ *all assuming* $0 \leq n < -\langle h_i, \lambda\rangle$.

(3.7)

Proof. This follows from (3.1)–(3.3). □

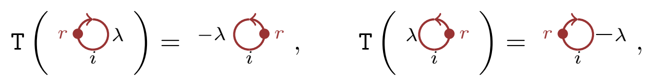
The dotted bubbles  ,  define endomorphisms of 1_λ for $r, s \geq 0$. We also give meaning to negatively dotted bubbles by making the following definitions for $r, s < 0$:



(3.8)

(3.9)

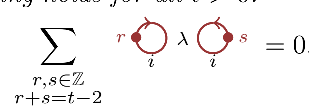
Note by Theorem 2.3, (2.1) and (3.4) that



(3.10)

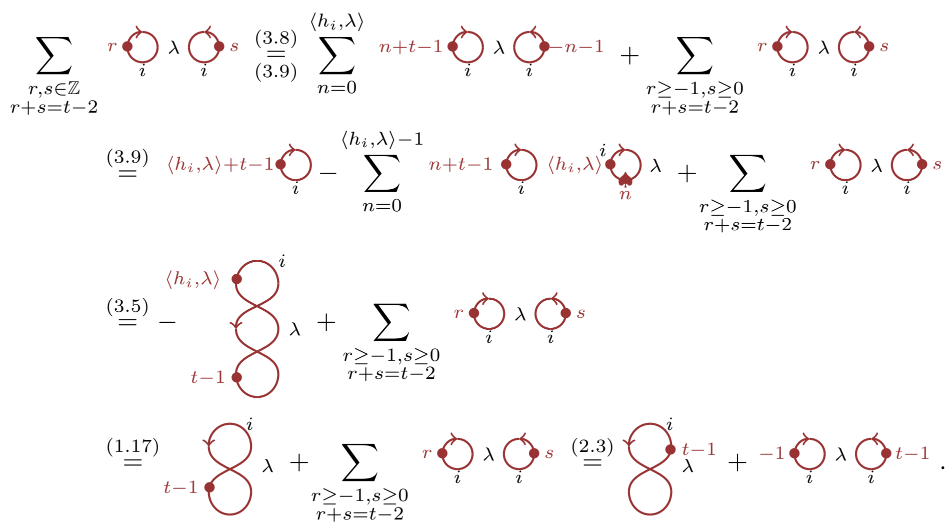
for all $r, s \in \mathbb{Z}$. Given (3.6)–(3.9), the following theorem implies the infinite Grassmannian relation as formulated in formulae (1.20)–(1.22) in the introduction.

Theorem 3.2. *The following holds for all $t > 0$:*

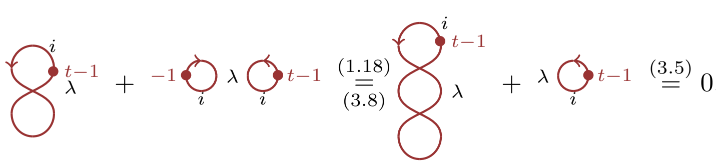


(3.11)

Proof. We prove this under the assumption that $\langle h_i, \lambda\rangle \geq 0$; the result when $\langle h_i, \lambda\rangle \leq 0$ then follows using Theorem 2.3 and (3.10). We have that

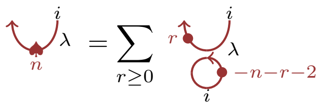


It just remains to show that the final expression here is zero. When $\langle h_i, \lambda\rangle > 0$ this follows by (3.6) and (3.8). Finally if $\langle h_i, \lambda\rangle = 0$ then

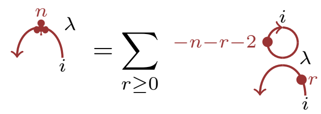


$\square$

Corollary 3.3. *The following relations hold:*

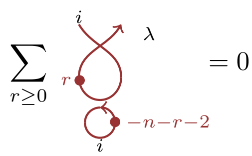


if $0 \le n < \langle h_i, \lambda\rangle$, (3.12)

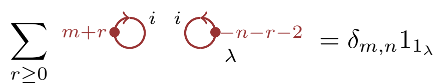


if $0 \le n < -\langle h_i, \lambda\rangle$. (3.13)

Proof. We explain the proof of (3.12); the proof of (3.13) is entirely similar. Remembering the definition (3.1), it suffices to show that the vertical composition consisting of (1.8) on top of (1.23) is equal to the identity. Using (3.5)–(3.6), this reduces to checking that

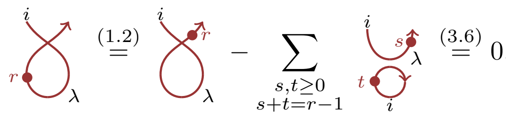


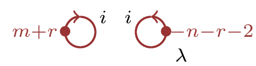
if $0 \le n < \langle h_i, \lambda\rangle$, (3.14)

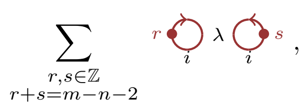


if $0 \le m, n < \langle h_i, \lambda\rangle$. (3.15)

For (3.14), each term in the summation is zero: if $r \ge \langle h_i, \lambda\rangle$ the counterclockwise dotted bubble is zero by (3.9); if $0 \le r < \langle h_i, \lambda\rangle$ we have that



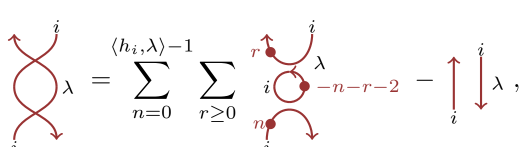
To prove (3.15), note by (3.6) and (3.9) that in order for  to be non-zero we must have $m+r \ge \langle h_i, \lambda\rangle - 1$ and $-n-r-2 \ge -\langle h_i, \lambda\rangle - 1$. Adding these inequalities implies that $m \ge n$. Moreover if $m = n$ the only non-zero term in the summation is the term with $r = \langle h_i, \lambda\rangle - 1 - m$, which equals 1_{1_λ} by (3.6) and (3.9) again. Finally if $m > n$ the left hand side of (3.15) can be rewritten as



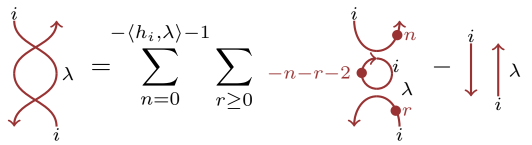
which is zero by (3.11). □

On substituting (3.12)–(3.13) into the definitions (3.1)–(3.2), this establishes the assertions about the 2-morphisms (1.23)–(1.24) made in the introduction.

Corollary 3.4. *The following relations hold:*

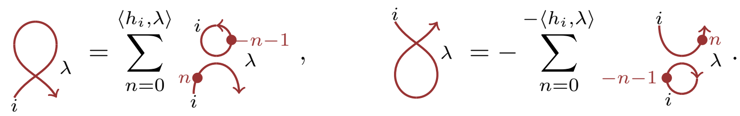


(3.16)

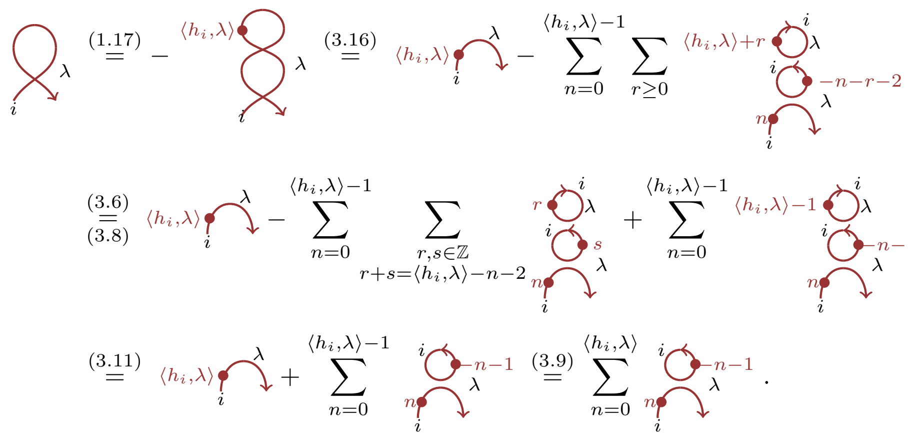

(3.17)

Proof. Substitute (3.12)–(3.13) into (3.5). □

Corollary 3.5. *The following relations hold:*


(3.18)

Proof. In view of Theorem 2.3, (2.1) and (3.10), it suffices to prove the left hand relation. We are done already by (3.7) if $\langle h_i, \lambda\rangle < 0$. If $\langle h_i, \lambda\rangle \geq 0$ then:

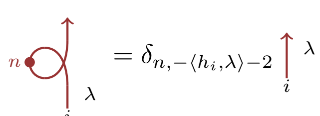


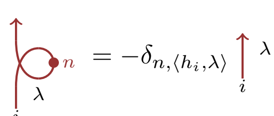
□

4. Left adjunction relations

In this section we show that the leftward cup and cap satisfy adjunction relations.

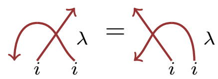
Lemma 4.1. *The following relations hold:*


if $0 \leq n \leq -\langle h_i, \lambda\rangle - 2,$ (4.1)

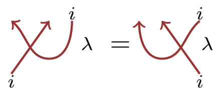

if $0 \leq n \leq \langle h_i, \lambda\rangle.$ (4.2)

Proof. Proceed by induction on n. For the base case, convert the upward crossings to rightward ones using (1.5)–(1.6), apply (3.18) and (3.8)–(3.9), then invoke (1.5). For the induction step, pull a dot past the crossing using (2.1) and (1.2), then use (3.6)–(3.7) and the induction hypothesis. □

Lemma 4.2. *The following relations hold:*

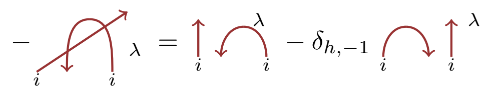


if $\langle h_i, \lambda\rangle < 0$, (4.3)



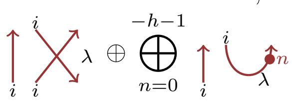
if $\langle h_i, \lambda\rangle > -2$. (4.4)

Proof. Let $h := \langle h_i, \lambda\rangle$ for short. First we prove (4.3), so $h < 0$. We claim that

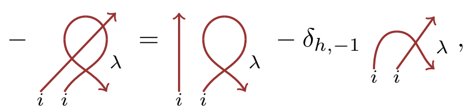


(4.5)

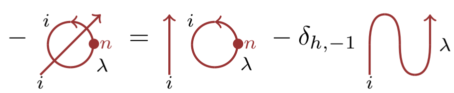
To establish the claim, we vertically compose on the bottom with the isomorphism



arising from (1.9) to reduce to showing equivalently that

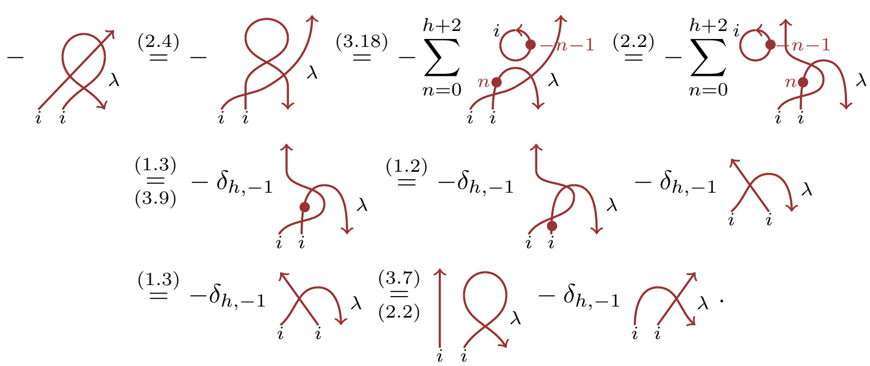


(4.6)

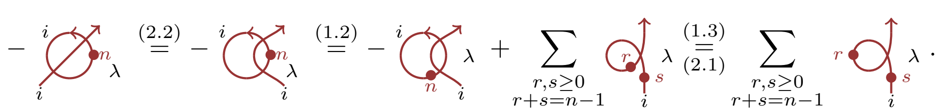


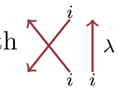
for $0 \leq n \leq -h-1$. (4.7)

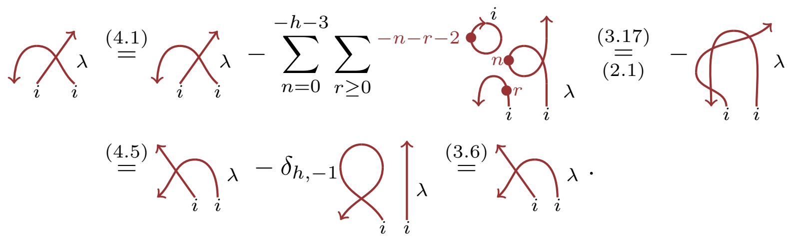
Here is the verification of (4.6):



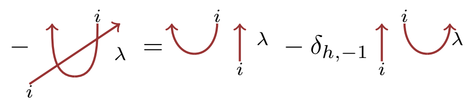
For (4.7), we note by (3.7) and (1.5) that the right hand side is equal to $\uparrow_i \lambda$ if $n = -h-1 > 0$, and it is zero otherwise. Now we simplify the left hand side:

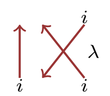


This is obviously zero if $n = 0$. Assuming $n > 0$, we apply (4.1) to see that it is zero unless $n = -h-1$, when the term with $r = -h-2, s = 0$ contributes $\uparrow_i \lambda$. This completes the proof of the claim. Now to establish the relation (4.3), we vertically compose (4.5) on the bottom with  to obtain the desired relation:

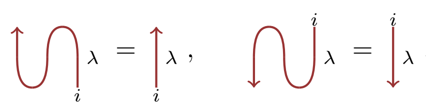


The proof of (4.4) follows by a very similar argument; one first checks that

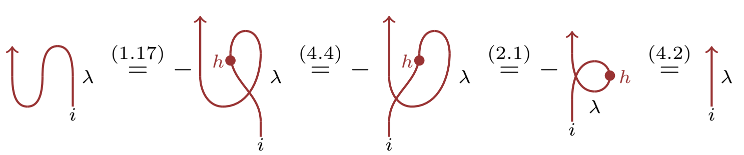


when $h > -2$ then vertically composes on the top with  . □

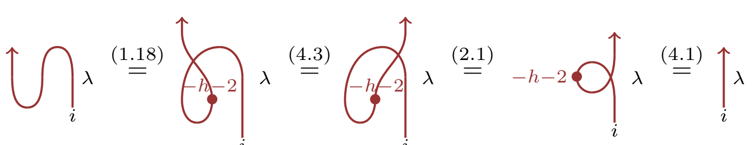
Theorem 4.3. *The following relations hold:*

 (4.8)

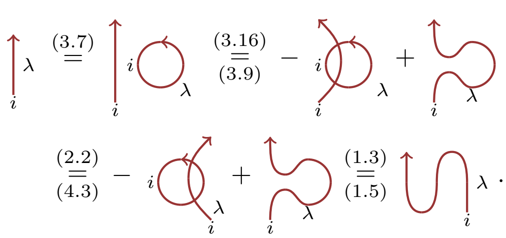
Proof. It suffices to prove the left hand relation; the right hand one then follows using Theorem 2.3. Let $h := \langle h_i, \lambda\rangle$ for short. If $h \geq 0$ then



If $h \leq -2$ then



Finally if $h = -1$ then

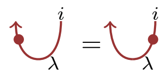


This completes the proof. □

5. Cyclicity relations

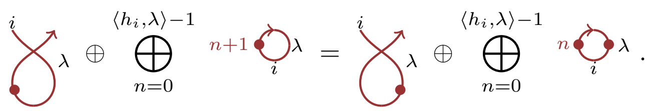
At this point, the proof of the Main Theorem is reduced to checking the cyclicity relations. We proceed to do this.

Lemma 5.1. *The following relation holds:*

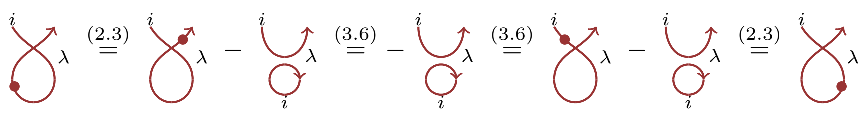


if $\langle h_i, \lambda\rangle \geq 1.$ (5.1)

Proof. The 2-morphisms on both sides of the desired identity go from 1_λ to $E_iF_i1_\lambda$. To show that they are equal, we vertically compose them both on the top with the isomorphism (1.8) to reduce to proving instead that

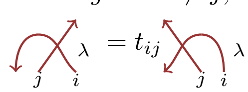


In these two column vectors of 2-morphisms, the entries involving dotted circles are equal thanks to (2.1). It just remains to observe that

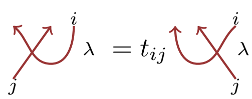


□

Lemma 5.2. *Assuming that* $i \neq j$*, the following relations hold:*

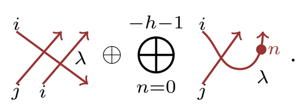


if $\langle h_i, \lambda\rangle < d_{ij},$ (5.2)

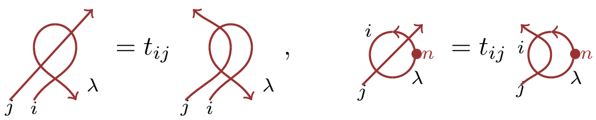


if $\langle h_i, \lambda\rangle \geq d_{ij}.$ (5.3)

Proof. Let $h := \langle h_i, \lambda\rangle$ for short. First we prove (5.2) assuming that $h \leq 0$. By (1.7) and (1.9), the following 2-morphism is invertible:

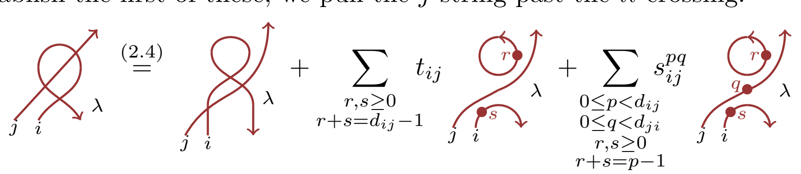


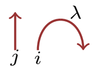
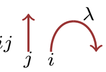
Vertically composing with this on the bottom, we deduce that the relation we are trying to prove is equivalent to the following relations:

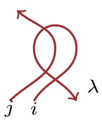
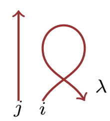
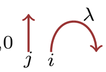


for $0 \leq n < -h.$ (5.4)

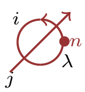
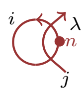
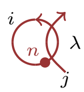
To establish the first of these, we pull the j-string past the ii-crossing:

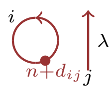
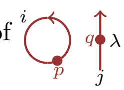


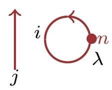
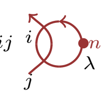
If $h < 0$ then all the terms on the right hand side vanish thanks to (3.7). If $h = 0$ and $d_{ij} > 0$ everything except for the $r = d_{ij} - 1$ term from the first sum vanishes, and we get t_{ij} . Finally if $h = d_{ij} = 0$, we only have the first term on the right hand side, which contributes t_{ij}  again thanks to (3.18), (3.9), (2.2) and (1.3). This is what we want because:

 $\overset{(1.19)}{=}$  $\underset{(3.9)}{\overset{(3.18)}{=}} \delta_{h,0}$ .

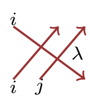
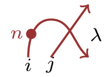
We are just left with the right hand relations from (5.4) involving bubbles:

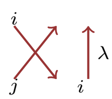
 $\overset{(2.2)}{=}$  $\overset{(1.2)}{=}$ 

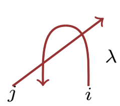
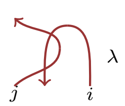
$\overset{(1.3)}{=} t_{ij}$  $+ \Big($ a linear comb. of  with $n \leq p < n + d_{ij} \Big)$

$\overset{(3.7)}{=} \delta_{n,-h-1}\ t_{ij}$  $\overset{(3.7)}{=} t_{ij}$  $\overset{(1.19)}{=} t_{ij}$ .

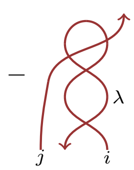
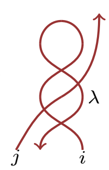
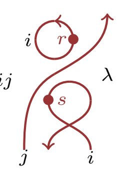
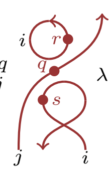
The relation (5.3) follows by a very similar argument to the one explained in the previous paragraph; the first step is to vertically compose on the top with the isomorphism

 $\oplus \bigoplus_{n=0}^{h-d_{ij}-1}$ .

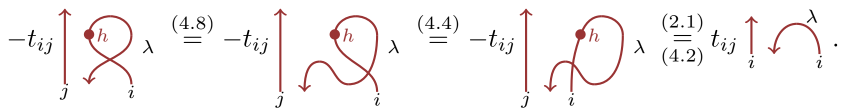
Finally we must prove (5.2) if $0 < h < d_{ij}$. We vertically compose on the bottom with the isomorphism  to reduce to proving that

 $= t_{ij}$ . (5.5)

To see this we apply (3.17) to transform the left hand side into

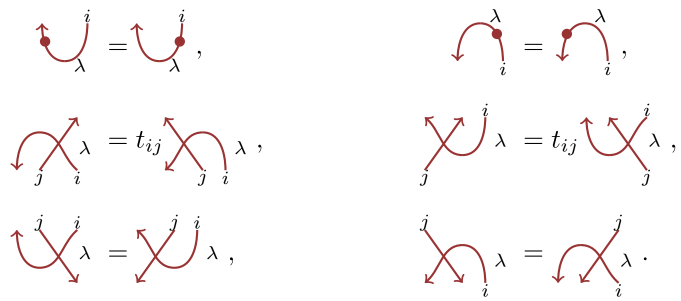
$-$  $\overset{(2.4)}{=} -$  $- \sum_{\substack{r,s\geq 0\\ r+s=d_{ij}-1}} t_{ij}$  $- \sum_{\substack{0\leq p<d_{ij}\\ 0\leq q<d_{ji}\\ r,s\geq 0\\ r+s=p-1}} s_{ij}^{pq}$ .

The first term on the right hand side here vanishes by (3.6). Also the terms in the summations are zero unless $r \geq d_{ij} - h - 1$ and $s \geq h$ by (3.6)–(3.7), hence we are left just with the $r = d_{ij} - h - 1, s = h$ term, which equals



This is equal to the right hand side of (5.5) thanks to (1.19). ☐

Theorem 5.3. *The following relations hold for all $i, j \in I$ and $\lambda \in P$:*



(5.6)

(5.7)

(5.8)

Proof. For (5.6), we already proved the left hand relation when $\langle h_i, \lambda \rangle \geq 1$ in (5.1). Now take this relation with λ replaced by $\lambda + \alpha_i$, attach leftward caps to the top left and top right strands, then apply (4.8) to prove the right hand relation when $\langle h_i, \lambda \rangle \geq -1$. Finally apply Theorem 2.3 to the cases established so far to get the right hand relation when $\langle h_i, \lambda \rangle \leq -1$ and the left hand relation when $\langle h_i, \lambda \rangle \leq 1$.

The proofs of (5.7)–(5.8) follow by a similar strategy to the previous paragraph, starting from (4.3)–(4.4) and (5.2)–(5.3). ☐

The final set of relations (1.26) needed to complete the proof of the Main Theorem follow easily from (5.6)–(5.8) using also (4.8).

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[L] A. Lauda, A categorification of quantum $\mathfrak{sl}(2)$, *Adv. Math.* **225** (2010), 3327–3424.

[R] R. Rouquier, 2-Kac-Moody algebras; `arXiv:0812.5023`.

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