WHAT IS MY RESEARCH ALL ABOUT?

JONATHAN BRUNDAN

Let me start with the idea of a polynomial algebra on n generators, like $\mathbb{R}[x_1, \dots, x_n]$. This consists of polynomials in the variables $x_1, \dots, x_n$, for instance:

$$p = 4x_1^3 + x_2x_3 - 5x_1^2x_4.$$

The symbol $\mathbb{R}$ indicates that all the coefficients in sight must be real numbers. Of course you can add two polynomials together and multiply them using the usual rules of arithmetic. This means that $\mathbb{R}[x_1, \dots, x_n]$ is an *algebra*, a sort of mathematical structure in which addition and multiplication makes sense.

Now let S_n be the *symmetric group* of all permutations of the integers 1 through n. What does this mean? For definiteness take $n = 4$. Then a permutation of $\{1, 2, 3, 4\}$ is a function $g : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ that is a 1–1 correspondence. For example the function g with $g(1) = 2, g(2) = 3, g(3) = 4$ and $g(4) = 1$ is a permutation of $\{1, 2, 3, 4\}$. It is usually denoted by the notation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

All you really have to do to get a permutation in S_4 is rearrange the numbers $1, 2, 3, 4$ in some order and write them into the second row of the above array, so there are $4! = 24$ different permutations in the group S_4.

The set S_n of permutations just introduced is called a *group* because there is a rule called *composition of functions*, denoted $\circ$, to multiply two permutations to get a third. For instance:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

To see what is going on here, think what the permutations do to the number 1. The second permutation on the left hand side of the equation sends 1 to 4. Then the first permutation sends 4 to 1. So overall 1 has gone to 1, which is the same as under the permutation on the right hand side. The same thing holds for the numbers 2, 3 and 4, hence the equality. Incidentally the permutation on the right hand side is called the *identity permutation*, because it does nothing. The two permutations on the left hand side are *inverses* because their product is the identity. A basic requirement in any group is that all the elements have inverses like this.

If p is a polynomial in $\mathbb{R}[x_1, \dots, x_n]$ and g is a permutation in S_n, you can replace each occurrence of x_1 in p by $x_{g(1)}$, each occurrence of x_2 by $x_{g(2)}$, and so on, to get a new polynomial denoted $g \cdot p$. For example if g and p are as in the examples displayed above, then

$$g \cdot p = 4x_2^3 + x_3x_4 - 5x_2^2x_1.$$

If a polynomial p has the property that $g \cdot p = p$ *for every* permutation g, then the polynomial p is called a *symmetric polynomial.* For example the following are symmetric polynomials in four variables:

$$\begin{aligned}
e_1 &= x_1 + x_2 + x_3 + x_4,\\
e_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,\\
e_3 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,\\
e_4 &= x_1x_2x_3x_4.
\end{aligned}$$

They are symmetric polynomials because they do not change if you permute the variables x_1, x_2, x_3 and x_4 around in any way you like. The above symmetric polynomials are very special and are called the *elementary symmetric polynomials.* You can build other symmetric polynomials out of them, for example

$$e_1e_4 = x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2$$

is another symmetric polynomial, as is any polynomial built by adding and multiplying e_1, e_2, e_3 and e_4 in any way you like.

We will write $\mathbb{R}[x_1, \dots, x_n]^{S_n}$ for the set of all symmetric polynomials in n variables. It is a *subalgebra* of $\mathbb{R}[x_1, \dots, x_n]$, because if you add or multiply two symmetric polynomials the result is again symmetric, hence the subset $\mathbb{R}[x_1, \dots, x_n]^{S_n}$ is actually itself an algebra with addition and multiplication operations in its own right. Now I can state one of the most classical theorems in all of algebra, dating back to the first half of the 19th century:

> *The subalgebra $\mathbb{R}[x_1, \dots, x_n]^{S_n}$ consisting of all symmetric polynomials in $\mathbb{R}[x_1, \dots, x_n]$ is itself a polynomial algebra with generators $e_1, \dots, e_n$.*

In particular this means that *any* symmetric polynomial can be built from the elementary symmetric polynomials by adding and multiplying.

This is a good start but we still haven't passed the year 1850! Now I want to introduce $n \times n$ matrices into the discussion. Let $\mathfrak{g}$ denote the Lie algebra of all $n \times n$ matrices with entries in $\mathbb{R}$. An element of $\mathfrak{g}$ is an $n \times n$ array of real numbers with rows numbered $1, \dots, n$ from top to bottom and columns numbered $1, \dots, n$ from left to right. For example, for i and j between 1 and n, we have the *ij-matrix unit* $e_{i,j}$, which is the matrix in $\mathfrak{g}$ with entry 1 in row i and column j, and zeros in all other entries. The matrix units generate $\mathfrak{g}$ in the sense that any other matrix can be obtained by adding together scalar multiples of matrix units. For example the following four matrix units generate $\mathfrak{g}$ in the case $n = 2$:

$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Most undergraduate students know the rule to multiply two matrices x and y together to get a new matrix $x \circ y$. It is easy to write this rule down for matrix units:

$$e_{i,j} \circ e_{k,l} = \begin{cases} e_{i,l} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Because the matrix units generate $\mathfrak{g}$, you can recover the rule to multiply arbitrary matrices from this formula.

I called $\mathfrak{g}$ a *Lie algebra* above; the name 'Lie' here honors the Norwegian mathematician Sophus Lie (1842–1899). This means that we are working with another multiplication on $\mathfrak{g}$ different from matrix multiplication, namely, the *matrix commutator* $[x, y]$ defined by $[x, y] = x \circ y - y \circ x$. The Lie algebra $\mathfrak{g}$ is intimately related to the *general linear group* G of all $n \times n$ invertible matrices under matrix multiplication. The elements of G are the $n \times n$ matrices g which possess an inverse, i.e. there is another matrix denoted g^{-1} such that the products $g \circ g^{-1}$ and $g^{-1} \circ g$ are both equal to the *identity matrix*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(Remember here that possessing inverses was the basic requirement for a group.) The general linear group G is one of the most fundamental examples of a *Lie group*.

We are going to be interested next in a certain polynomial algebra denoted $S(\mathfrak{g})$. The elements of $S(\mathfrak{g})$ are polynomials just like elements of $\mathbb{R}[x_1, \dots, x_n]$, but in the variables $e_{i,j}$ $(i, j = 1, \dots, n)$ instead of in the variables $x_1, \dots, x_n$. For example,

$$p = e_{1,1}e_{2,2} - e_{1,2}$$

is a polynomial in $S(\mathfrak{g})$. Given an element g in G and a polynomial p in $S(\mathfrak{g})$, we obtain a new polynomial $g \cdot p$ by replacing each occurrence of $e_{i,j}$ in p by the matrix $g \circ e_{i,j} \circ g^{-1}$, the *conjugate* of $e_{i,j}$ by g. When we considered the action of the group S_n on the polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$ above, we considered the subalgebra $\mathbb{R}[x_1, \dots, x_n]^{S_n}$ of symmetric polynomials. In our new situation, we have the group G acting on the polynomial algebra $S(\mathfrak{g})$, so again we can consider the *invariant subalgebra* $S(\mathfrak{g})^G$ consisting of all polynomials p in $S(\mathfrak{g})$ that satisfy $g \cdot p = p$ for every g in G. The next important theorem, which was already known in Lie's lifetime, is as follows:

> *The subalgebra $S(\mathfrak{g})^G$ of $S(\mathfrak{g})$ is a polynomial algebra on n generators.*

For example if $n = 2$ then $S(\mathfrak{g})^G$ is a polynomial algebra with the following two generators:

$$e_{1,1} + e_{2,2}, \qquad e_{1,1}e_{2,2} - e_{2,1}e_{1,2}.$$

In particular, every element of $S(\mathfrak{g})^G$ can be built by adding and multiplying these two special invariants.

The two theorems formulated so far are actually intimately related, as follows. Define a function

$$\pi : S(\mathfrak{g}) \to \mathbb{R}[x_1, \dots, x_n]$$

by replacing each $e_{i,j}$ in a polynomial in $S(\mathfrak{g})$ by x_i if $i = j$ or by 0 if $i \neq j$. The following fundamental theorem is a special case of the much more general *Chevalley restriction theorem* proved in the 1940s:

> *The restriction of the map π defines an isomorphism*
>
> $$S(\mathfrak{g})^G \to \mathbb{R}[x_1, \dots, x_n]^{S_n}$$
>
> *between the invariant subalgebra of $S(\mathfrak{g})$ and the algebra of symmetric polynomials.*

For example, the two generators for $S(\mathfrak{g})^G$ in the case $n = 2$ displayed in the previous paragraph map under the function π to the polynomials

$$x_1 + x_2, \qquad x_1 x_2,$$

which are just the two elementary symmetric functions e_1, e_2 that generate $\mathbb{R}[x_1, x_2]^{S_2}$. It is not obvious for $n > 2$ how exactly to write down n elements $f_1, \dots, f_n$ in $S(\mathfrak{g})^G$ which map under the isomorphism π to the elementary symmetric functions $e_1, \dots, e_n$ in $\mathbb{R}[x_1, \dots, x_n]^{S_n}$. However a beautiful formula giving $f_1, \dots, f_n$ explicitly was already known in the 19th century: the ith elementary invariant f_i is the coefficient of x^{n-i} in the expansion of the determinant

$$\det \begin{pmatrix} x + e_{1,1} & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & x + e_{2,2} & \cdots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & x + e_{n,n} \end{pmatrix}.$$

The polynomial algebra $S(\mathfrak{g})$ is *commutative*, which means that $xy = yx$ for any two elements x, y in $\mathfrak{g}$; in other words,

$$xy - yx = 0$$

always. There is a closely related algebra denoted $U(\mathfrak{g})$, the *universal enveloping algebra* of $\mathfrak{g}$. The elements of $U(\mathfrak{g})$ again look like polynomials in the matrix units $e_{i,j}$, but the multiplication is no longer quite commutative: instead we have that

$$xy - yx = [x, y]$$

for x and y in $\mathfrak{g}$, where $[x, y]$ is the matrix commutator from before. The resulting algebra $U(\mathfrak{g})$ is a much more interesting algebra than $S(\mathfrak{g})$. Since it is not commutative, it is sensible to try to describe its *center* $Z(\mathfrak{g})$, which means all elements p in $U(\mathfrak{g})$ with the property that $px = xp$ for all matrices x in $\mathfrak{g}$. The following theorem is a special case of *Harish-Chandra's theorem* describing the center of the universal enveloping algebra of an arbitrary semisimple Lie algebra:

> *The center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ is a polynomial algebra on n generators.*

For example if $n = 2$, it turns out that the elements

$$e_{1,1} + e_{2,2} - 1, \qquad e_{1,1}(e_{2,2} - 1) - e_{2,1}e_{1,2}$$

generate $Z(\mathfrak{g})$. These are almost the same as the elements that generated $S(\mathfrak{g})^G$ described before, but because things are no longer commutative it becomes quite subtle: we've had to replace $e_{2,2}$ by $e_{2,2} - 1$ everywhere in order to get elements of $Z(\mathfrak{g})$ out of the elements of $S(\mathfrak{g})^G$ listed before.

Finally I can explain a joint theorem with my student Jonathan Brown, also proved independently by Panyushev, Premet and Yakimova. Take any matrix e in $\mathfrak{g}$. For example if $n = 3$ we could take the matrix

$$e = e_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The *centralizer* of e in $\mathfrak{g}$ means all matrices x in $\mathfrak{g}$ that commute with e, i.e. that satisfy $[x, e] = 0$. It is usually denoted by $\mathfrak{g}_e$, and it is a Lie subalgebra of $\mathfrak{g}$. In the above example, $\mathfrak{g}_e$ is generated by the matrices

$$e_{1,1} + e_{2,2}, \quad e_{1,2}, \quad e_{3,3}, \quad e_{1,3}, \quad e_{3,2}.$$

We proved the following:

The center $Z(\mathfrak{g}_e)$ of the universal enveloping algebra $U(\mathfrak{g}_e)$ is a polynomial algebra on n generators.

In fact we gave a precise determinantal formula giving a set of n explicit generators, rather like but much more complicated than the determinantal formula described on the previous page. In the $n = 3$ case with $e = e_{1,2}$ as above, our formula gives the following three generators for $Z(\mathfrak{g}_e)$:

$$e_{1,1} + e_{2,2} + e_{3,3} - 1, \quad e_{1,2}, \quad e_{1,2}(e_{3,3} - 1) - e_{3,2}e_{1,3}.$$

If one takes $e = 0$ then $\mathfrak{g}_e = \mathfrak{g}$ and our theorem specializes to Harish-Chandra's theorem. Thus we have generalized one of the most classical results in Lie theory to centralizers of matrices. However the methods we used to do this are far from classical, relying on some quite new algebras called *shifted Yangians*, which I discovered in some joint work a couple of years ago with my collaborator Alexander Kleshchev.