Homework 4, due March 13

1. Let $E \subset \mathbb{P}^2$ be a smooth *elliptic* curve cut out by the equation $y^2z - x^3 - axz^2 - bz^3 = 0$ (in particular, the polynomial $t^3 + at + b$ does not have multiple roots). We denote $e := (0,1,0) \in E$. We consider a map $\varphi\colon E \to \mathrm{Pic}^0(E),\ u \to [u - e]$. Prove ~~that~~ part (c)
(we proved a), b), d) in class)
a) φ is injective.
b) Given $u, v \in E$, there exists $w \in E$ such that $[u + v] = [w + e] \in \mathrm{Pic}^2(E)$.
c) For any effective divisor D on E of degree d, there is $w \in E$ such that

$$[D] = [w + (d-1)e] \in \mathrm{Pic}^d(E).$$

d) φ is a bijection of sets $E \xrightarrow{\sim} \mathrm{Pic}^0(E)$. It is used to equip $\mathrm{Pic}^0(E)$ with a structure of algebraic curve, and to equip E with a structure of algebraic group.

2. a) Prove that $\dim \Gamma(E, \mathcal{O}_E(ke)) = k$ for any $k > 0$.
b) Write down all the elements of $\Gamma(E, \mathcal{O}_E(ke))$ explicitly in the form $P(x) + yQ(x)$ for some $P, Q \in \mathsf{k}(x)$.

3. a) If $u + v = w$ in the sense of the group law of Problem 1 on E, write down the coordinates of w in terms of coordinates of u, v.
b) Prove that $u + v + w = 0$ iff u, v, w lie on a line $\mathbb{P}^1 \subset \mathbb{P}^2$.

4. Prove that if $u + v = 0$, and u (resp. v) has coordinates $(x_1, y_1, 1)$ (resp. $(x_2, y_2, 1)$), then $x_1 = x_2,\ y_1 = -y_2$.

5. A point $u \in E$ is called an *inflection point* if the tangent line ℓ_u at u has the intersection order 3 (as opposed to the usual order 2) with E at u (in particular, $\ell_u \cap E = \{u\}$). Prove that
a) u is an inflection point on E iff u has order 3 in the group law of Problem 1 on E.
b) If a line $\ell \subset \mathbb{P}^2$ passes through two inflection points $u_1, u_2 \in E$, then the third point of $\ell \cap E$ is an inflection point as well.