Homework 3, due Feb. 21

1. Consider a cubic curve $C_a \subset \mathbb{P}^2$ cut out by the equation
$$x^3 + y^3 + z^3 + a(x + y + z)^3 = 0.$$
a) Find all a such that C_a is singular.
b) Find all the singular points of C_a.
c) For which a the curve C_a is irreducible?
d) Prove that if a cubic curve $C \subset \mathbb{P}^2$ has 3 singular points, then C is a union of 3 lines.

e) Prove that if a cubic curve $C \subset \mathbb{P}^2$ is irreducible and not smooth, then it is birationally isomorphic to $\mathbb{P}^1$.

2. Prove that if a degree $d \geq 2$ hypersurface $X \subset \mathbb{P}^n$ contains a linear subspace $L \simeq \mathbb{P}^r \subset \mathbb{P}^n$, $r \geq n/2$, then X can not be smooth.

3. a) Let $X \subset \mathbb{P}(V)$ be a smooth hypersurface of degree $d \geq 2$. Prove that the set of all hyperplanes $H \subset \mathbb{P}(V)$ tangent to X at some (varying) point $x \in X$ forms a hypersurface $X^\vee \subset \mathbb{P}(V^*)$. In case X is singular, we define $X^\vee$ as the closure of the set of hyperplanes tangent to X at the smooth points of X.
b) Determine the dual curve $X^\vee$ for the curve $X \subset \mathbb{P}^2$ cut out by the equation
$$x^3 + y^3 + z^3 = 0.$$

4. Find all the singular points of the a) *Steiner surface* cut out by the equation
$$x_1^2 x_2^2 + x_0^2 x_2^2 + x_0^2 x_1^2 - x_0 x_1 x_2 x_3 = 0.$$
b) *Dual Steiner surface* cut out by the equation
$$y_0 y_1 y_2 + y_0 y_1 y_3 + y_0 y_2 y_3 + y_1 y_2 y_3 = 0.$$

5. Assume k has charactertic 0. a) Prove that almost all (that is, all but finitely many) fibers of a function $f : \mathbb{A}^n \to \mathbb{A}^1$ are smooth.
b) Prove that for a morphism $\pi : X \to Y$ of smooth algebraic varieties, there is a nonempty open subset $U \subset Y$ such that for any $y \in U$ the fiber $\pi^{-1}(y)$ is smooth.
c) Prove the following stronger version of Bertini theorem. Let $X \subset \mathbb{P}^n$ be a smooth irreducible subvariety, and let $L \simeq \mathbb{P}^{n-2} \subset \mathbb{P}^n$ be a linear subspace of codimension 2 such that $L \cap X$ is smooth and L is not contained in X. Consider the *Lefschetz pencil* $P \simeq \mathbb{P}^1$ formed by all the hyperplanes $L \subset H \subset \mathbb{P}^n$. Then for almost all $H \in P$, the intersection $X \cap H$ is smooth.