Homework 2, due Feb. 2

optional

1. Let $\mathcal{F}$ be a coherent sheaf on an open subvariety $U \subset X$ of an algebraic variety. Prove that there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U = \mathcal{F}$.

2. a) Prove that the Grassmannian $\mathrm{Gr}(2,4)$ can be embedded into $\mathbb{P}^5$ as a quadratic hypersurface Π (*Plücker quadric*). Thus the points of the Plücker quadric parametrize the lines $\mathbb{P}^1 \subset \mathbb{P}^3$.

b) Prove that any plane $\mathbb{P}^2 \subset \Pi \subset \mathbb{P}^5$ is of the following kind: it is either the set of lines $\ell \subset \mathbb{P}^3$ containing a point $p \in \mathbb{P}^3$, or the set of lines $\ell \subset \mathbb{P}^3$ contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$.

3. Let $X \subset \mathbb{P}^3$ be a surface of degree d cut out by a homogeneous polynomial F (thus X is represented by a point of $\mathbb{P}^N$, $N = \frac{1}{6}(d+1)(d+2)(d+3) - 1$, given by F up to proportionality). Prove that

a) the incidence subset $\Gamma = \{(\ell, F) \ : \ \ell \subset X\} \subset \Pi \times \mathbb{P}^N$ is actually a closed subvariety (cut out by equations homogeneous in coefficients of F and in Plücker coordinates on Π).

b) Γ is irreducible.

c) $\dim \Gamma = \frac{1}{6}d(d+1)(d+5) + 3$.

d) If $d > 3$, then a surface X corresponding to a point of an open (nonempty) subset $U \subset \mathbb{P}^N$ contains no lines.

4. Prove that a) the cubic surface $X_0 \subset \mathbb{P}^3$ cut out by the equation $x_1x_2x_3 = x_0^3$ contains exactly three lines.

b) Any cubic surface $X \subset \mathbb{P}^3$ contains at least one line.

c) There is a nonempty open subset $U \subset \mathbb{P}^{19}$ parametrizing cubic surfaces such that any cubic surface corresponding to a point of U contains finitely many lines.

d) There are cubic surfaces containing infinitely many lines.

5. Let $F_0(x_0, \ldots, x_n), \ldots, F_n(x_0, \ldots, x_n)$ be homogeneous polynomials of degrees $d_0, \ldots, d_n$. Here $x_0, \ldots, x_n$ are coordinates on an $n+1$-dimensional vector space V. Let $\Gamma \subset \mathbb{P}(V) \times \prod_{i=0}^n \mathbb{P}(\mathrm{Sym}^{d_i} V^*)$ be the closed subvariety formed by all the collections $(\underline{x}, F_0, \ldots, F_n)$ such that $F_0(\underline{x}) = \ldots = F_n(\underline{x}) = 0$. Let $\varphi \colon \Gamma \to \prod_{i=0}^n \mathbb{P}(\mathrm{Sym}^{d_i} V^*)$ denote the projection. Prove that

a) $\dim \Gamma + 1 = \dim \varphi(\Gamma) + 1 = \dim \prod_{i=0}^n \mathbb{P}(\mathrm{Sym}^{d_i} V^*)$.

b) There exists a polynomial $R(F_0, \ldots, F_n)$ in coefficients of F_i such that $R = 0$ iff the system $F_0 = \ldots = F_n = 0$ has a nonzero solution.

6. Let $X \subset \mathbb{P}^n$ be a hypersurface cut out by a homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree d.

a) Prove that the singular points of X are the solutions of the system of equations $F = \frac{\partial F}{\partial x_0} = \ldots = \frac{\partial F}{\partial x_n} = 0$.

b) Prove that if d is not divisible by the characteristic of k, then the first equation $F = 0$ in a) follows from the other ones.

c) Assume k has charactertic 0. Prove that the set S of all $F \in \mathbb{P}(\mathrm{Sym}^d V^*)$ defining singular hypersurfaces in $\mathbb{P}^n$ forms a hypersurface $S \subset \mathbb{P}(\mathrm{Sym}^d V^*)$.