Homework 1, due Jan. 19

1. Let $f \in \mathsf{k}[x_1, \ldots, x_n]$. Prove $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

2. Let X and Y be affine algebraic varieties. Prove

$$\Omega[X \times Y] = \Omega[X] \otimes_{\mathsf{k}[X]} \mathsf{k}[X \times Y] \oplus \Omega[Y] \otimes_{\mathsf{k}[Y]} \mathsf{k}[X \times Y].$$

3. Let $C \subset \mathbb{A}^2$ be the affine algebraic curve cut out by $x^2 + y^2 = 1$. Let $\operatorname{char} \mathsf{k} \neq 2$. Prove that

a) Any regular function on C can be uniquely written in the form $f(y) + xg(y)$ for some polynomials f, g.

b) Any regular differential form on C can be uniquely written in the form $f(y)dx + (g(y) + xh)dy$ for some polynomials f, g and a constant $h \in \mathsf{k}$.

c) The differential forms $\frac{dx}{y}$ on D_y and $-\frac{dy}{x}$ on D_x coincide on the intersection D_{xy} of these open sets. Thus they give rise to a global regular differential form ω on C.

d) Write down ω in the form of b).

4. Consider the elliptic curve $E \subset \mathbb{P}^2$ cut out by $y^2 z = x(x - z)(x - \lambda z)$, where $\lambda \in \mathsf{k}$ is a fixed parameter distinct from 0 and 1. In the open chart $D_z \cap E$, consider the differential form $\omega = \frac{dx}{y}$. Prove that

a) ω is regular on $D_z \cap E$.

b) ω extends (uniquely) to a regular differential form on the whole of E.

c) ω vanishes nowhere on E.

d) Any global regular differential form on E is a scalar multiple of ω.

5. Prove that any algebraic group is smooth at any point.

6. a) Let $C \subset \mathbb{A}^3$ be a reducible algebraic curve equal to the union of 3 coordinate lines. Prove that the ideal $J_C \subset \mathsf{k}[x, y, z]$ can not be generated by 2 elements.

b) Give an example of an affine curve that can not be a closed subvariety in $\mathbb{A}^{2021}$.

c) Give an example of an irreducible affine curve that can not be a closed subvariety in $\mathbb{A}^{2021}$.

optional